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Introduction to Axiomatic Geometry

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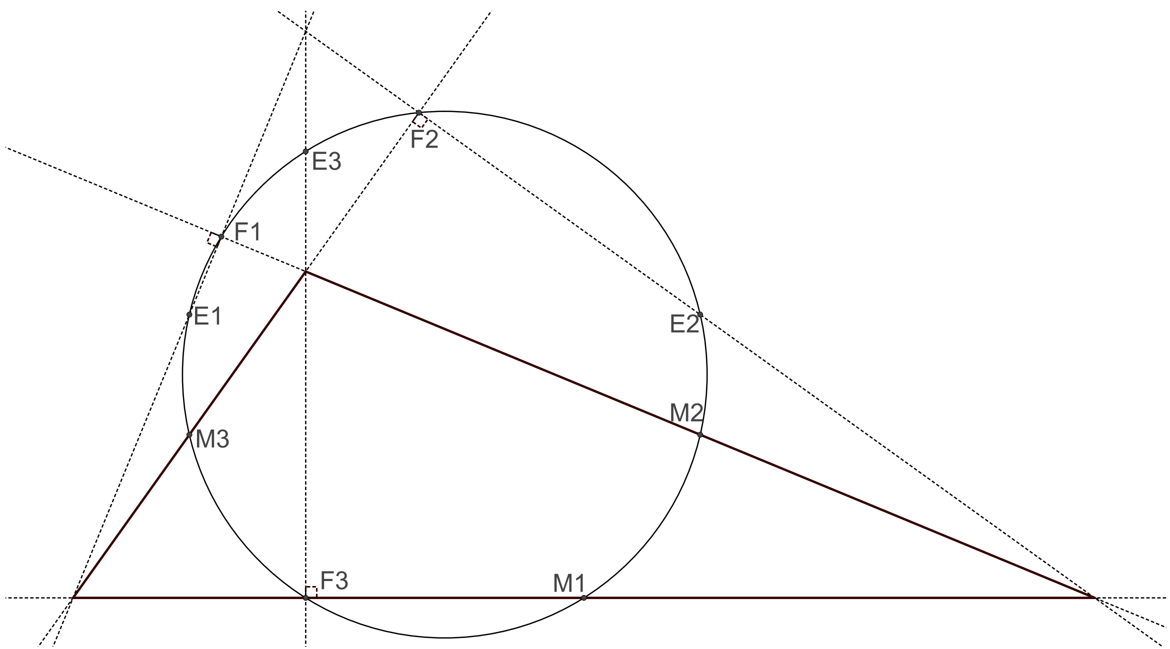

a text for a Junior-Senior Level College Course in Introduction to Proofs and Euclidean Geometry

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Introduction to Axiomatic Geometry

Preface

This book presents Euclidean Geometry and was designed for a one-semester course preparing junior and senior level college students to teach high school Geometry. (I have used it many times for a 3000-level Geometry course at Ohio University in Athens.)

The book could also serve as a text for a sophomore or junior level Introduction to Proofs course. Axiom systems are introduced at the beginning of the book, and throughout the book there is a lot of discussion of how one structures a proof. Care is taken to discuss the idea of negating quantified statements, and to discuss the significance of the converse and contrapositive of conditional statements. Methods of indirect proof are introduced, both the usual method of contradiction and the simpler (and usually overlooked) method of proving the contrapositive.

The book also discusses extensively what it means to *use* an old theorem in a proof of a new theorem. That is, the student must make clear what prior steps in the current proof confirm that the hypotheses of the old theorem are satisfied.

Some portions of the text and some of the exercises are flagged as *Advanced*. This material can be omitted, or the Theorems can be accepted without proof. Students who are up for the challenge can read those sections and work on those exercises.

The Definitions and Theorems are numbered, and complete lists of them are presented in the Appendices. In the print version of the book, the Appendices are bound in a separate cover. When I use the book in my course, students are continually asked to justify claims (claims that the students make or that are made in the book) by referring to the lists of Definitions and Theorems. Students are even allowed to use the lists on quizzes and exams. This instills in them the concept of a sequence of Definitions and Theorems that builds on earlier results, and it helps them really learn the list.

The axiom system includes the existence of a distance function, coordinate functions, and an angle measurement function. This is not the approach of Hilbert (and of books that use Hilbert's approach), in which the behavior of points on a line is dictated by Axioms of Betweenness, and then the existence of a distance function can proven as a theorem. I have two reasons for choosing to include the existence of a distance function and coordinate functions as an axiom, rather than proving them as in theorems. First, the axioms used in high school Geometry books usually include the existence of a distance function and coordinate functions as an axiom, so this book will more closely align with what students will need to be able to teach in the future.

A second reason for choosing to include the existence of a distance function and coordinate functions as an axiom is that I feel that is important to include a variety of proof styles. In this book, the proofs involving the Plane Separation Axiom are mostly about logic: arguments often boil down to recognizing that the Plane Separation Axiom includes conditional statements, and that these conditional statements come with associated contrapositive statements that are also true. In books that also contain Axioms of Betweenness, one finds a lot more proofs, about behavior of points on lines, that are mostly about logic. I don't think that students need to study that so many proofs of that style. In this book, by contrast, the theorems about the behavior of points on lines have proofs that involve the properties of functions. That is because in this book, the behavior of points on lines is entirely determined by the existence of coordinate functions, and the definition of a coordinate function involves the idea of one-to-one and onto functions. The proofs of theorems about the behavior of points on lines are of a style very different from the style of the proofs of theorems involving plane separation. In addition to learning to do proofs of a different style, the students benefit from having to work with the definitions of one-to-one and onto functions, concepts that are underrepresented in prerequisite courses.

It is significant that the axiom system does not include any axioms about area. The approach is as follows. Triangle Similarity for Euclidean Geometry is developed in theorems. One of the last result of similarity is a theorem that states that for a Euclidean triangle, the product of base times height does not depend on the choice of base. This enables the definition of triangle area in terms of base times height. Then the area of more complicated "polygonal regions" is possible by subidividing those regions into triangles. Properties of area are proven in theorems. (In the final section of the chapter on area, this approach to area is contrasted to the approach taken by high school geometry books, where area and area properties are included in the axioms.)

Drawings play a large role in the exposition. Because this is not a commercial textbook, page space is not precious. As a result, it is feasible to include many drawings to illustrate proofs that might be given only a single drawing in a commercial book. In some of the proofs, a new drawing is made for each step of the proof. When teaching the course, I have the students work on the skill of making a new, separate, drawing to illustrate each step of their proofs.

Throughout the book, the writing is meant to have a level of precision appropriate for a junior or senior level college math course. The language of quantifiers, conditional statements, functions and their properties, and function notation are all used. At the end of three chapters, there are short sections that compare the presentation of the material in this book to the presentation that students will encounter in a high school book, where quantifiers and conditional statements are not so clear and where function notation is not used. It is pointed out that a number of theorems that have difficult proofs are given as axioms in high school Geometry books.

Each chapter of the book ends with exercises that are organized by section.

Throughout the PDF version of the book, most references are actually hyperlinks. That is, any reference to a numbered book section, or numbered theorem, can be clicked on to take the reader to see that numbered item. Using the "back arrow" will take the reader back to where they were before.

The book was designed to work for a one-semester course. Therefore, much geometry is not included. It is important to be clear about what this book *does not* contain. Here is a short list.

The book does not yet have an index. That will be fixed soon. Meanwhile, the reader of the electronic version of the book can, of course, search for a term and turn up all occurrences, including the definition (if there is one).

Three dimensional objects are not discussed and are not mentioned in the axioms.

Constructions are not included. This is partly because I did not want the book to be too full. But also, I feel that constructions amount to using an alternate axiom system, the axioms that describe what can be constructed with the basic tools. A “construction” is really a proof of existence using this alternate axiom system. In many books that I have read, there is not a good distinction between the idea of a construction—using the alternate axiom system—and the idea of an existence proof—using the regular axiom system. For example, in some books, authors will write, in the course of an existence proof, that some object with certain properties can be “constructed”. What should really be written is that some object with certain properties can be “proven to exist”. For this book, I decided to stick to one axiom system and omit constructions.

Spherical Geometry is not included. I plan to include a bit of spherical geometry in future versions of the book, but not very much. I will include some exercises in the early chapters that explore the fact that spherical geometry flunks the earliest of the axioms in this book, the *Incidence Axioms*. In this sense, Spherical Geometry is very different from Euclidean Geometry. Because of this, I feel that studying Spherical Geometry will not shed much useful light on Euclidean Geometry. That is why I have, for now, omitted it.

Hyperbolic Geometry is not included. I hope to include a substantial amount of Hyperbolic Geometry in a future version, but when I do, it will be for a second semester course. However, the book in its current form is organized in a way that makes it easy to mention the idea of Hyperbolic Geometry during the course, and I do this quite frequently when I teach. The axiom system consists of ten “Neutral Geometry” axioms, numbered <N1> through <N10> and an eleventh “Euclidean Parallel Axiom”, <EPA>. Chapters three through eight deal only with Neutral Geometry—using only the first ten axioms. When I teach the course, I often introduce two informal “drawn models” of Neutral Geometry. One is the usual “Euclidean” model where straight-looking drawn lines play the role of the undefined lines in the geometry. The second model is the Poincare disk model, where drawn circular arcs play the role of the undefined lines in the geometry. I frequently ask students to produce both a straight-line illustration and Poincare disk illustration for concepts discussed in the first eight chapters. This makes it clear that there are (at least) two flavors of Neutral Geometry. Chapters nine through fifteen study the full Euclidean Geometry, with all eleven axioms. Students see that the eleventh axiom, the Euclidean Parallel Axiom, really is needed to nail down all of the behavior that one expects in Euclidean Geometry. (Remark: One of my students went on to study Hyperbolic Geometry in an independent study course with me, using a book about Hyperbolic Geometry. She was constantly referring to chapters three through eight of my book for Neutral Geometry theorems and proofs that are used in Hyperbolic Geometry.)

The final chapter of the book discusses Maps, Transformations, and Isometries. This chapter includes much discussion of functions, inverse functions, maps of the plane, and even includes a review of binary operations and groups. It culminates in a proof that the set of isometries of the plane is a group, and also has some theorems about classifying isometries. That final chapter *is not* about a “transformational approach” to Euclidean geometry. I hope to include chapters on a transformational approach in future versions of the book.

I welcome feedback and suggestions.

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1.Axiom Systems

1.1. Introduction to Axiom Systems

1.1.1. Looking Back at Earlier Proofs Courses

In earlier courses involving proofs, you studied conditional statements—statements of the form

If Statement A is true then Statement B is true.

You saw that the proof of such a conditional statement has the following form:

Proof:
(1) Statement *A* is true. (given)
(2) Some statement (with some justification provided)
(3) Some statement (with some justification provided)
(4) Some statement (with some justification provided)
(5) Statement *B* is true. (with some justification provided)
End of Proof

For example, you proved the following conditional statement about integers:

If n *is odd, then* n^2 *is odd.*

Proof:
(1) Suppose that n is an odd integer.
(2) There exists some integer k such that $n = 2k + 1$ (by statement (1) and the definition of odd)
(3) $n^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ (arithmetic)
(4) $(2k^2 + 2k)$ is an integer (because the set of integers is closed under addition and multiplication). Call this integer m. So, $n = 2m + 1$ where m is an integer.
(5) n^2 is odd. (by statement (4) and the definition of odd)
End of Proof

In this proof, the only assumption is the given Statement *A*. The proof does not prove that Statement *A* is true; it only proves that *if* statement *A is* true, then statement *B* is also true.

But in most of your proofs in those earlier courses, certainly in all of the proofs involving basic number theory, there were actually unstated assumptions in addition to the explicitly given Statement *A*. The proof above relies on some unstated assumptions about the basic properties of integers. These basic properties of integers are not statements that can be proven. Rather, they are statements that are *assumed* to be true. Statements such as

The set of integers is closed under addition and multiplication

or

There is a special number called 1 with the following properties
1) 1 is not equal to 0.
2) For all integers n, $n \cdot 1 = n$.

It may seem silly that we are even discussing the above statements. They are so obviously true, and we've known that they are true since grade school math. We never think about the statements, even though we do integer arithmetic all the time.

Well, the statements can not in fact be proven *true* (or *false*). We can't imagine what we would do if they were false, though: all of the math we've used since grade school would be out the window. So we just assume that all the basic properties of the integers are true and we go from there. But to be totally honest about all the given information in the Theorem about odd numbers, we would need to write the theorem and its proof something like this:

> *Theorem: If all of the axioms of the integers are true [they would have to be listed explicitly here] and* n *is odd, then* n^2 *is also odd.*

Proof:
(1) Suppose that all of the axioms of the integers are true [they would have to be listed explicitly here] and that n is an odd integer.
(2) There exists some integer k such that $n = 2k + 1$ (by statement (1) and the definition of odd)
(3) $n^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ (arithmetic)
(4) $(2k^2 + 2k)$ is an integer (because the set of integers is closed under addition and multiplication)
(5) n^2 is odd. (by statement (4) and the definition of odd)
End of Proof

But in your earlier courses about proofs, the emphasis was on learning how to build a proof. That is, you would have been concerned with steps 2 through 5 of the above proof. Since we all know how to work with the integers without having to refer to their underlying axioms, the axioms were not mentioned.

1.1.2. Looking Forward

Having studied in earlier courses the building of proofs of basic number theory theorems, a setting where the underlying axioms could go unmentioned, you will study in this book the building of proofs of theorems about geometry. In geometry, the underlying axioms are not obvious and cannot go unmentioned. In fact, this book will largely be about the axioms, themselves. We will start by studying very simple axioms systems, in order to learn how one builds proofs based on an explicit axiom system and to learn some of the basic terminology of axiom systems. Then, we will turn our attention to more complicated axiom systems.

1.1.3. Definition of Axiom System

We will use the term *axiom system* to mean a finite list of statements that are assumed to be true. The individual statements are the *axioms*. The word *postulate* is often used instead of *axiom*. In this book, axioms will be labeled with numbers enclosed in angle brackets: <1>, <2>, etc.

Example 1 of an axiom system

<1> Elvis is dead.
<2> Chocolate is the best flavor of ice cream.
<3> $5 = 7$.

Notice that the first statement is one that most people are used to thinking of as true. The second sentence is clearly a statement, but one would not have much luck trying to find general agreement as to whether it is true or false. But if we list it as an axiom, we are assuming it is true.

The third statement seems to be problematic. If we insist that the normal rules of arithmetic must hold, then this statement could not possibly be true. There are two important issues here. The first is that if we are going to insist that the normal rules of arithmetic must hold, then that means our axiom system is actually larger than just the three statements listed: the axiom system would also include the axioms for arithmetic. The second issue is that if we do assume that the normal rules of arithmetic must hold, and yet we insist on putting this statement on the list of axioms, then we have a "bad" axiom system in the sense that its statements contradict each other. We will return to this when we discuss *consistency* of axiom systems.

So the idea is that regardless of whether or not we are used to thinking of some statement as true or false, when we put the statement on a list of axioms we are simply assuming that the statement is true.

With that in mind, we could create a slightly different axiom system by modifying our first example.

Example 2 of an axiom system

<1> Elvis is alive.
<2> Chocolate is the best flavor of ice cream.
<3> $5 = 7$.

The statements of an axiom system are used in conjunction with the rules of inference to prove theorems. Used this way, the axioms are actually part of the hypotheses of each theorem proved. For example, suppose that we were using the axiom system from Example 2, and we were somehow able to use the rules of inference to prove the following theorem from the axioms.

Theorem: If Bob is Blue then Ann is Red.

Then what we really would have proven is the following statement:

Theorem: If ((Elvis is alive) and (Chocolate is the best flavor of ice cream) and ($5 = 7$*) and (Bob is Blue)) then Ann is Red.*

1.1.4. Primitive Relations and Primitive Terms

As you can see from the examples in the previous section, axiom systems may be comprised of statements that we are used to thinking of as true, or statements that we are used to thinking of as false, or some mixture of the two. More interestingly, an axiom system can be made up of statements whose truth we have no way of assessing. The easiest way to get such an axiom system is to build statements using words whose meaning has not been defined. In this course, we will be doing this in two ways.

1.1.5. Primitive Relations

The first way of building statements whose meaning is undefined is to use nouns whose meaning is known in conjunction with transitive verbs whose meaning is not known. For instance, consider the following sentence about two integers.

"5 *is related to* 7."

This is a sentence with the noun 5 as the subject, the noun 7 as the direct object, and the words "*is related to*" as the transitive verb. We have no idea what this sentence might mean, because the phrase *"is related to"* is undefined. That is, the transitive verb is undefined.

Transitive verbs in the written language have a counterpart in the mathematical language: they correspond to mathematical *relations*. The undefined phrase *"is related to"* in the previous paragraph is an example of an *undefined relation on the set of integers*. In the context of axiom systems, an *undefined relation* is sometimes called a *primitive relation*. When one presents an axiom system that contains primitive relations—that is, undefined transitive verbs—it is important to introduce those primitive relations before listing the axioms. Here is an example of an axiom system consisting of sentences built using primitive relations in the manner described above.

Axiom System:	Axiom System #1
Primitive Relations:	relation on the set of integers spoken "*x* is related to *y*"
Axioms:	<1> 5 is related to 7 <2> 5 is related to 8 <3> For all integers x and y, if x is related to y, then y is related to x. <4> For all integers x, y, and z, if x is related to y and y is related to z, then x is related to z.

We can easily abbreviate the presentation of this axiom system by using the symbols and terminology of mathematical relations:

- The set of integers is denoted by the symbol $\mathbb{Z}$. (This kind of font is often called "double-struck" or "blackboard bold". That is, the symbol $\mathbb{Z}$ is a double-struck Z, or a blackboard-bold Z.)
- The undefined relation is denoted by the symbol $\mathcal{R}$. (script R)
- The relation $\mathcal{R}$ is a *relation on the set of integers*. From your previous study of relations, you should have learned that this means simply that $\mathcal{R}$ is a subset: $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$. Because the relation $\mathcal{R}$ is undefined, we don't know what that subset is.
- The symbol ${}_5\mathcal{R}_7$ is used as an abbreviation for the sentence "5 *is related to* 7", which means that the ordered pair $(5,7)$ is an element of the subset $\mathcal{R}$. That is, $(5,7) \in \mathcal{R}$.
- More generally, the symbol ${}_x\mathcal{R}_y$ is used as an abbreviation for *"x is related to y"*.
- A relation on a set is said to be *symmetric* if it has the following property:

  For all elements a and b in the set, if a is related to b, then b is related to a.

We see that axiom <3> is simply saying that relation $\mathcal{R}$ is symmetric.

- A relation on a set is said to be *transitive* if it has the following property:

  For all elements a, b, and c in the set, if a is related to b, and b is related to c, then a is related to c.

  We see that axiom <4> is simply saying that relation $\mathcal{R}$ is transitive.

Using the symbols and terminology described above, the abbreviated version of Axiom System #1 is as follows.

Axiom System:	Axiom System #1, abbreviated version
Primitive Relations:	relation $\mathcal{R}$ on the set $\mathbb{Z}$, spoken "x is related to y"
Axioms:	<1> ${}_5\mathcal{R}_7$ <2> ${}_5\mathcal{R}_8$ <3> relation $\mathcal{R}$ is symmetric <4> relation $\mathcal{R}$ is transitive

Observe that each of the axioms is a statement whose truth we have no way of assessing, because the relation $\mathcal{R}$ is undefined. But we can prove the following theorem.

Theorem for axiom system #1: 7 is related to 8.

Proof

(1) 7 is related to 5 (by axioms <1> and <3>)

(2) 7 is related to 8 (by statement (1) and axioms <2> and <4>)

End of proof

As mentioned in the previous section, the axioms could be stated explicitly as part of the statement of the theorem. (Then we would not really need to state the axiom system separately.)

Theorem: If (($\mathcal{R}$ is a relation on $\mathbb{Z}$) and (${}_5\mathcal{R}_7$) and (${}_5\mathcal{R}_8$) and ($\mathcal{R}$ is symmetric) and ($\mathcal{R}$ is transitive), then ${}_7\mathcal{R}_8$.

1.1.6. Primitive Terms

The second way of building statements whose meaning is undefined is to use not only undefined transitive verbs, but also undefined nouns. A straightforward way to do this is to introduce sets A and B whose elements are undefined. For instance, let A be the set of *akes* and B be the set of *bems*, where *ake* and *bem* are undefined nouns. Introduce the following sentence: "*the ake is related to the bem*". Note that this is a sentence with the undefined noun *ake* as the subject, the words "*is related to*" as the transitive verb, and the undefined noun *bem* as the direct object. Of course we have no idea what this sentence might mean, because the nouns *ake* and *bem* are undefined. But we now have the following building blocks that can be used to build sentences.

- the undefined noun: *ake*
- the undefined noun: *bem*

- the undefined sentence: *The ake is related to the bem.*

Since we don't know the meaning of the sentence "*The ake is related to the bem*", we have effectively introduced an undefined relation from set A to set B. We could call this undefined relation $\mathcal{R}$. Using the standard notation for relations, we could write $\mathcal{R} \subset A \times B$. The sentence "*The ake is related to the bem*" would be denoted by the symbol ${}_{ake}\mathcal{R}_{bem}$. and would mean that the ordered pair(ake, bem) is an element of the subset $\mathcal{R}$. That is, $(ake, bem) \in \mathcal{R}$.

In the context of axiom systems, an *undefined noun* is sometimes called an *undefined term*, or a *primitive term*, or an *undefined object*, or a *primitive object*. In presentations of axiom systems that contain primitive terms, the primitive terms are customarily listed along with the primitive relations, before the axioms. Here is an example of an axiom system consisting of sentences built using primitive terms and primitive relations in the manner described above.

Axiom System:	Axiom System #2
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from A the set of all *akes* to B the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> For any *bem*, there are exactly two *akes* that are related to the *bem*.

As with Axiom System #1, each of these axioms in Axiom System #2 is a statement whose truth we have no way of assessing, because the words *ake* and *bem* are undefined and the relation is undefined. But we can prove the following theorem.

Theorem #1 for Axiom System #2: There are exactly 6 *bems*.

Proof

Part 1: Show that there must be at least 6 *bems*.

(1) By axiom <1>, there are four *akes*. Therefore, it is possible to build six unique sets of two *akes*. Those six sets are $\{ake_1, ake_2\}$, $\{ake_1, ake_3\}$, $\{ake_1, ake_4\}$, $\{ake_2, ake_3\}$, $\{ake_2, ake_4\}$, $\{ake_3, ake_4\}$.

(2) Consider the set $\{ake_1, ake_2\}$. Axiom <2> tells us that there must be a *bem* that both of these *akes* are related to. Call it bem_1.

(3) Now consider the set $\{ake_1, ake_3\}$. Axiom <2> tells us that there must be a *bem* that both of these *akes* are related to. Axiom <3> tells us that it cannot be bem_1, because there are already two *akes* that are related to bem_1. So there must be a new *bem* that ake_1 and ake_3 are both related to. Call it bem_2.

(4) Proceeding this way, we see that there must be at least 6 *bems*, one for each of the sets of two *akes* listed above.

Part 2: Show that there cannot be more than 6 bems. (indirect proof)

(5) Suppose that there is a 7th *bem*, called bem_7. (assumption for indirect proof)

(6) By axiom <3>, there must be exactly two *akes* that are related to bem_7. Let those two akes be denoted ake_j and ake_k, where $j \neq k$. From Part 1 of this proof, we know that these two *akes* are also both related to one of the *bems*

numbered bem_1, bem_2, …, bem_6. So there are two *bems* that ake_j and ake_k are both related to.

(7) Statement (6) contradicts Axiom <2>. Therefore, our assumption in statement (5) was incorrect. There cannot be a 7th *bem*.

End of proof

As with the theorem that we proved in the previous section for Axiom System #1, we note that the theorem just presented could be written with all of the primitive terms, primitive relations, and axioms put into the hypothesis. The resulting theorem statement would be quite long.

Theorem: If *Blah Blah Blah* then there are exactly 6 *bems*.

In the exercises, you will prove the following:

Theorem #2 for Axiom System #2: For every *ake*, there are exactly three *bems* that the *ake* is related to.

Digression to Discuss Proof Structure

Before going further, it is worthwhile to pause and consider the structure of the proofs of Axiom System #2 Theorems #1 and #2.

Note that Theorem #1 is an *existential statement*: it states that something *exists*. In order to prove the theorem, one must use the axioms. Now consider the three axioms of Axiom System #2. Notice that axioms <2> and <3> say something about objects existing, but only in situations where some other prerequisite objects are already known to exist. Those axioms are of no use to us until after we have proven that those other prerequisite objects do exist. Only axiom <1> says simply that something exists, with no prerequisites. So we have no choice but to start the proof of Theorem #1 by using axiom <1>. Look back at the proof of Theorem #1, and observe that it does start by using axiom <1>.

Now note that Theorem #2 starts with the words "For every *ake*...". A statement that begins this way is called a *universal statement*. It claims that each ake in the set of all akes has the stated property. (In this case, the property is that there are exactly three *bems* that the *ake* is related to.) The set of all *akes* could be considered a sort of universal set in this situation, with Theorem #2 making a claim about every *ake* in that universal set. Hence, the name *universal statement*.

It is important to keep in mind that the universal statement of Theorem #2 does not claim that any *akes* exist. It only makes a claim about an *ake* that is already known to exist: an *ake* that is *given*. So a proof of Theorem #2 must start with a sentence introducing a *given ake*. The given *ake* is known to have only the attributes mentioned before the comma in the "for every..." phrase. In the statement of Theorem #2, the phrase simply says *"For every ake,..."*. That is, the given *ake* is not known to have any other attributes beyond merely exiting. So The start of the proof would look like this:

Start of proof of Theorem #2

(1) Suppose that an *ake* is given. (Notice that no other attributes are mentioned.)

(2)

(The goal of the proof will be to prove that the given *ake* does in fact have another attribute, he property is that there are exactly three *bems* that the *ake* is related to)

Let me reiterate that one does not begin the proof of Theorem #2 by proving that an *ake* exists. Even though Axiom System #2 does have an axiom that states that four *akes* exist, the statement of Theorem #2 does not claim that any *akes* exist. Theorem #2 only makes a claim about a *given ake*, and so the proof of Theorem #2 must start with the introduction of a *given ake*.

More generally, in the proof of a universal statement, one must start the proof by stating that a *generic object* is given. A *generic object* is an object or objects that have only the attributes mentioned right after the words "for all" and before the comma in the theorem statement. The given objects are known to have those attributes, but are not known to have any other. (The object of the proof is to prove that the given objects do have some other attributes as well.)

End of Digression to Discuss Proof Structure

Our examples of axiom systems with undefined terms and undefined relations seem rather absurd, because their axioms are meaningless. What purpose could such abstract collections of nonsense sentences possibly serve? Well, the idea is that we will use such abstract axiom systems to represent actual situations that are not so abstract. Then for any abstract theorem that we have been able to prove about the abstract axiom system, there will be a corresponding true statement that can be made about the actual situation that the axiom system is supposed to represent.

This begs the question: why study the axiom system at all, if the end goal is to be able to prove statements that are about some actual situation? Why not just study the actual situation and prove the statements in that context? The answer to that is twofold. First, a given abstract axiom system can be recycled, used to represent many different actual situations. By simply proving theorems once, in the context of the axiom system, the theorems don't need to be reproved in each actual context. Second, and more important, by proving theorems in the context of the abstract axiom system, we draw attention to the fact that the theorems are true by the simple fact of the axioms and the rules of logic, and nothing else. This will be very important to keep in mind when studying axiomatic geometry.

1.1.7. Interpretations and Models

As mentioned above, an axiom system with undefined terms and undefined relations is often used to represent an actual situation. This idea of representation is made more precise in the following definition. You'll notice that the representing sort of gets turned around: we think of the actual situation as a representation of the axiom system.

Definition 1 Interpretation of an axiom system

Suppose that an axiom system consists of the following four things

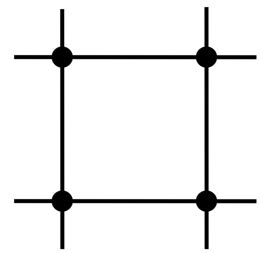
- an undefined object of one type, and a set A containing all of the objects of that type
- an undefined object of another type, and a set B containing all of the objects of that type
- an undefined relation $\mathcal{R}$ from set A to set B
- a list of axioms involving the primitive objects and the relation

An interpretation of the axiom systems is the following three things

- a designation of an actual set A' that will play the role of set A
- a designation of an actual set B' that will play the role of set B
- a designation of an actual relation $\mathcal{R}'$ from A' to B' that will play the role of the relation $\mathcal{R}$

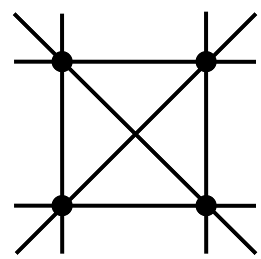
As examples, for Axiom System #2 from the previous section we will investigate three different interpretations invented by Alice, Bob, and Carol. Recall Axiom System #2 included the following things

- an undefined term *ake* and a set A containing all the *akes*
- an undefined term *bem* and a set B containing all the *bems*
- a primitive relation $\mathcal{R}$ from set A to set B
- a list of three axioms involving these undefined terms and the undefined relation



Alice's interpretation of Axiom System #2.

- Let A' be the set of dots in the picture at right.
- Let B' be the set of segments in the picture at right.
- Let relation $\mathcal{R}'$ from A' to B' be defined by saying that the words *"the ake is related to the segment"* mean *"the dot touches the segment"*.



Bob's interpretation of Axiom System #2.

- Let A' be the set of dots in the picture at right.
- Let B' be the set of segments in the picture at right.
- Let relation $\mathcal{R}'$ from A' to B' be defined by saying that the words *"the dot is related to the segment"* mean *"the dot touches the segment"*.

Carol's interpretation of Axiom System #2.

- Let A' be the set whose elements are the letters v, w, x, and y. That is, $A' = \{v, w, x, y\}$.
- Let B' be the set whose elements are the sets $\{v, w\}$, $\{v, x\}$, $\{v, y\}$, $\{w, x\}$, $\{w, y\}$, and $\{x, y\}$. That is, $B' = \{\{v, w\}, \{v, x\}, \{v, y\}, \{w, x\}, \{w, y\}, \{x, y\}\}$.
- Let relation $\mathcal{R}'$ from A' to B' be defined by saying that the words *"the letter is related to the set"* mean *"the letter is an element of the set"*.

Notice that Alice and Bob have slightly different interepretations of the axiom system. Is one better than the other? It turns out that we will consider one to be much better than the other. The criterion that we will use is to consider what happens when we translate the Axioms into statements about dots and segments. Using a find & replace feature in a word processor, we can simply replace every occurrence of ake with dot, every occurrence of bem with $segment$, and every occurrence of *is related to* with *touches*. Here are the resulting three statements.

Statements:

1.There are four *dots*. These may be denoted *dot*$_1$, *dot*$_2$, *dot*$_3$, *dot*$_4$.
2. For any two distinct *dots*, there is exactly one *line* that both *dots* touch.
3. For any *line*, there are exactly two *dots* that touch the *line*.

We see that in Bob's interpretation, all three of these statements are true. In Alice's interpretation the first and third statements are true, but the second statement is false.

What about Carol's interpretation? We should consider what happens when we translate the Axioms into statements about letters and sets. We can simply replace every occurrence of *ake* with *letter*, every occurrence of *bem* with *set*, and every occurrence of *is related to* with *is an element of*.

Statements:

1. There are four *letters*. These may be denoted $letter_1$, $letter_2$, $letter_3$, $letter_4$.
2. For any two distinct *letters*, there is exactly one *set* that both *letters* are elements of.
3. For any *set*, there are exactly two *letters* that are elements of the *set*.

We see that in Carol's interpretation, the translations of the three axioms are three statements that are all true.

Let's formalize these ideas with a definition.

Definition 2 successful interpretation of an axiom system; model of an axiom system
To say that an interpretation of an axiom system is *successful* means that when the undefined terms and undefined relations in the axioms are replaced with the corresponding terms and relations of the interpretation, the resulting statements are all true. A *model* of an axiom system is an *interpretation* that is *successful*.

So we would say that Bob's and Carol's interpretations are successful: they are models. Alice's interpretation is unsuccessful: it is not a model.

Notice also that Bob's and Carol's models are essentially the same in the following sense: one could describe a correspondence between the objects and relations of Bob's model and the objects and relations of Carol's model in a way that all corresponding relationships are preserved. Here is one such correspondence.

objects in Bob's model	$\leftrightarrow$	objects in Carol's model
the lower left *dot*	$\leftrightarrow$	the letter v
the lower right *dot*	$\leftrightarrow$	the letter w
the upper right *dot*	$\leftrightarrow$	the letter x
the upper left *dot*	$\leftrightarrow$	the letter y
the *segment* on the bottom	$\leftrightarrow$	the set $\{v, w\}$
the *segment* that goes from lower left to upper right	$\leftrightarrow$	the set $\{v, x\}$
the *segment* on the left side	$\leftrightarrow$	the set $\{v, y\}$
the *segment* on the right side	$\leftrightarrow$	the set $\{w, x\}$
the *segment* that goes from upper left to lower right	$\leftrightarrow$	the set $\{w, y\}$
the *segment* across the top	$\leftrightarrow$	the set $\{x, y\}$
relation in Bob's model	$\leftrightarrow$	relation in Carol's model
the *dot* touches the *segment*	$\leftrightarrow$	the letter is an element of the set

What did I mean above by the phrase "*...in a way that all corresponding relationships are preserved...*"? Notice that the following statement is true in Bob's model.

The lower right *dot* touches the *segment* on the right side.

If we use the correspondence to translate the terms and relations from Bob's model into terms and relations from Carol's model, that statement becomes the following statement.

The letter w is an element of the set $\{w, x\}$.

This statement is true in Carol's model. In a similar way, any true statement about relationships between dots and segments in Bob's model will translate into a true statement about relationships between letters and sets in Carol's model.

The notion of two models being essentially the same, in the sense described above, is formalized in the following definition.

Definition 3 isomorphic models of an axiom system

Two models of an axiom system are said to be *isomorphic* if it is possible to describe a correspondence between the objects and relations of one model and the objects and relations of the other model in a way that all corresponding relationships are preserved.

It should be noted that it will not always be the case that two models for a given axiom system are isomorphic. We will return to this in the next section, when we discuss *completeness*.

But before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 1.4 on page 29.

1.2. Properties of Axiom Systems I: Consistency and Independence

In this section, we will discuss three important properties that an axiom system may or may not have. They are *consistency*, *completeness*, and *independence*.

1.2.1. Consistency

We will use the following definition of consistency.

Definition 4 consistent axiom system

An axiom system is said to be *consistent* if it is possible for all of the axioms to be true. The axiom system is said to be *inconsistent* if it is not possible for all of the axioms to be true.

We will be interested in determining if a given axiom system is consistent or inconsistent. It is worthwhile to think now about how one would prove that an axiom system is consistent, or how one would prove that an axiom system is inconsistent.

Suppose that one suspects that an axiom system is consistent and wants to prove that it is consistent. One proves that an axiom system is consistent by producing a model for the axiom system. For Axiom System #2, we have two models—Bob's and Carol's—so the axiom system is definitely consistent.

Suppose that one suspects that an axiom system is inconsistent and wants to prove that it is inconsistent. One would be trying to prove that something is not possible. It is not obvious how that would be done. The key is found in the Rule of Inference called the Contradiction Rule.

Digression to Consider Different Versions of the Contradiction Rule

We will consider the different versions of the Contradiction Rule. We will number the versions, so that we can more easily refer to them later. The Contradiction Rule is presented in lists of Rules of Inference in the following basic form:

Contradiction Rule Version 1 (Basic Form)	$\sim p \rightarrow c$ $\therefore p$

In this rule, the symbol c stands for a contradiction—a statement that is always false. This rule is used in the following way. To prove that statement p is true using the method of contradiction, one starts by assuming that statement p was false. One then shows that it is possible to reach a contradiction. Therefore, the statement p must be true.

Suppose we replace the statement p with the statement $\sim q$. Then the contradiction rule becomes

Contradiction Rule Version 2	$\sim(\sim q) \rightarrow c$ $\therefore (\sim q)$

In other words,

Contradiction Rule Version 2	$q \rightarrow c$ $\therefore \sim q$

This version of the rule states that if one can demonstrate that statement q leads to a contradiction, then statement q must be false.

There are other versions of this rule as well. Consider what happens if we use a statement q of the form $statement_1 \wedge statement_2$.

Contradiction Rule Version 3	$(statement_1 \wedge statement_2) \rightarrow c$ $\therefore \sim(statement_1 \wedge statement_2)$

If we apply DeMorgan's law to the conclusion of this version of the rule, we obtain

Contradiction Rule Contradiction Rule Version 3	$(statement_1 \wedge statement_2) \rightarrow c$ $\therefore \sim statement_1 \vee \sim statement_2$

This version of the rule says if one can demonstrate that an assumption that $statement_1$ and $statement_2$ are both true leads to a contradiction, then at least one of the statements must be false.

Finally, consider what happens if we use a whole list of statements.

Contradiction Rule Version 4	$(statement_1 \wedge statement_2 \wedge \cdots \wedge statement_k) \rightarrow c$ $\therefore \sim statement_1 \vee \sim statement_2 \vee \cdots \vee \sim statement_k$

This version of the rule states that if one can demonstrate that an assumption that a whole list of statements is true leads to a contradiction, then at least one of the statements must be false.

End of Digression to Consider Different Versions of the Contradiction Rule

Now return to the notion of an inconsistent axiom system. Recall in an inconsistent axiom system, it is impossible for all of the axioms to be true. In other words, at least one of the axioms must be false. We see that the Contradiction Rule Version 4 could be used to prove that an axiom system is inconsistent. That is, if one can demonstrate that an assumption that a whole list of axioms is true leads to a contradiction, then at least one of the axioms must be false.

We can create an example of an inconsistent axiom system by messing up Axiom system #2. We mess it up by appending a fourth axiom in a certain way.

Axiom System:	Axiom System #3, an example of an inconsistent axiom system
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from A the set of all *akes* to B the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> For any *bem*, there are exactly two *akes* that are related to the *bem*. <4> There is exactly one *bem*.

Using Axiom System #3, we can prove the following two theorems.

Theorem 1 for Axiom System #3: There are exactly 6 *bems*.

The proof is the exact same proof that was used to prove the identical theorem for axiom system #2. The proof only uses the first three axioms.

But the statement of Theorem #1 contradicts Axiom <4>! We have demonstrated that an assumption that the four axioms from Axiom System #3 are true leads to a contradiction. Therefore, at least one of the axioms must be false. In other words, Axiom System #3 is inconsistent.

Note that it is tempting to say that it must be axiom <4> that is false, because there was nothing wrong with the first three axioms before we threw in the fourth one. But in fact, there is not really anything wrong with the fourth axiom in particular. For example, if one discards axiom

<3>, then it turns out that the remaining list of axioms is perfectly consistent. Here is such an axiom system, with the axioms re-numbered. (You are asked to prove that this axiom system is consistent in Exercise [7] at the end of the chapter.)

Axiom System:	Axiom System #4, an example of a consistent axiom system
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from A the set of all *akes* to B the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> There is exactly one *bem*.

So the problem with Axiom system #3 is not with any one particular axiom. Rather, the problem is with the whole set of four axioms.

Before going on to read the next subsection, you should do the exercises for the current subsection. The exercises are found in Section 1.4 on page 29.

1.2.2. Independence

An axiom system that is *not consistent* could be thought of as one in which the axioms *don't agree*; an axiom system that is *consistent* could be thought of as one in which there is *no disagreement*. In this sort of informal language, we could say that the idea of *independence* of an axiom system has to do with whether or not there is any *redundancy* in the list of axioms. The following definitions will make this precise.

Definition 5 dependent and independent axioms

An axiom is said to be *dependent* if it is possible to prove that the axiom is true as a consequence of the other axioms. An axiom is said to be *independent* if it is not possible to prove that it is true as a consequence of the other axioms.

We will be interested in determining if a given axiom is dependent or independent. It is worthwhile to think now about how one would prove that an axiom is dependent, or how one would prove that an axiom is independent.

Suppose that one suspects that a given axiom is dependent and wants to prove that it is dependent. To do that, one proves that the statement of the axiom must be true with a proof that uses only the other axioms. That is, one stops assuming that the given axiom is a true statement and downgrades it to just an ordinary statement that might be true or false. If it is possible to prove the statement is true using a proof that uses only the other axioms, then the given axiom is dependent.

For an example of a *dependent* axiom, consider the following list of axioms that was constructed by appending an additional axiom to the list of axioms for Axiom System #2.

Axiom System:	Axiom System #5, containing a dependent axiom
Primitive Terms:	*ake*, *bem*

Primitive Relations:	relation from A the set of all *akes* to B the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> For any *bem*, there are exactly two *akes* that are related to the *bem*. <4> There are exactly six *bems*.

We recognize the first three axioms. They are the axioms from Axiom System #2. And we also recognize the statement of axiom <4>. It is the same statement as Theorem #1 for Axiom System #2. In other words, it can be proven that axiom <4> is true as a consequence of the first three axioms. So Axiom <4> is not independent; it is *dependent*.

Now suppose that one suspects that a given axiom is *independent* and wants to prove that it is independent. To do that, one stops assuming that the statement of the given axiom is true, and downgrades it to just an ordinary statement that might be true or false. One must produce two interpretations:

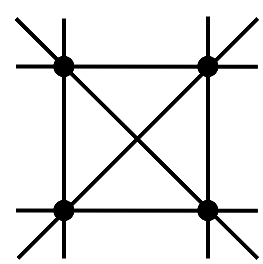
(1) One interpretation in which the statements of all of the other axioms are true and the statement of the given axiom is true. (That is, the statements of all the axioms are true.)
(2) A second interpretation, in which the statements of all of the other axioms are true and the statement of the given axiom is false.

For an example of an independent axiom, consider axiom <3> from Axiom System #2. (Refer to Axiom System #2 in Section 1.1.6 on page 15.) Axiom <3> is an independent axiom. To prove that it is independent, we stop assuming that the statement of axiom <3> is true, and downgrade it to just an ordinary statement that might be true or false.

Now consider two interpretations for Axiom System #2

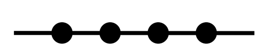
Bob's interpretation of Axiom System #2.

- Let A' be the set of dots in the picture at right.
- Let B' be the set of segments in the picture at right.
- Let relation $\mathcal{R}'$ from A' to B' be defined by saying that the words *"the dot is related to the segment*" mean *"the dot touches the segment"*.



Dan's interpretation of Axiom System #2.

- Let A' be the set of dots in the picture at right.
- Let B' be the set of segments in the picture at right.
- Let relation $\mathcal{R}'$ from A' to B' be defined by saying that the words *"the dot is related to the segment*" mean *"the dot touches the segment"*.



Consider the translation of the statement of axiom <3> into the language of the models:

- For any $segment$ there are exactly two *dots* that touch the *segment*.

We see that in Bob's interpretation of Axiom System #2, the statement of axiom <3> is *true*, while in Dan's interpretation of Axiom System #2, the statement of axiom <3> is *false*. So based on these two examples, we can say that in Axiom System #2, axiom <3> is independent.

The following definition is self-explanatory.

Definition 6 independent axiom system

An axiom system is said to be *independent* if all of its axioms are independent. An axiom system is said to be *not independent* if one or more of its axioms are not independent.

To prove that an axiom system is independent, one must prove that each one of its axioms is independent. That means that for each of the axioms, one must go through a process similar to the one that we went through above for Axiom <3> from Axiom System #2. This can be a huge task.

On the other hand, to prove that an axiom system is not independent, one need only prove that one of its axioms is not independent.

Before going on to read the next section, you should do the exercises for the current subsection. The exercises are found in Section 1.4 on page 29.

1.3. Properties of Axiom Systems II: Completeness

1.3.1. Completeness

Recall that in Section 1.1.7, we found that Bob's and Carol's models of Axiom System #2 were isomorphic models. It turns out that any two models for that axiom system are isomorphic. Such a claim can be rather hard—or impossible—to prove, but it is a very important claim. It essentially says that the axioms really "nail down" every aspect of the behavior of any model. This is the idea of *completeness*.

Definition 7 complete axiom system

An axiom system is said to be *complete* if any two models of the axiom system are isomorphic. An axiom system is said to be *not complete* if there exist two models that are not isomorphic.

It is natural to wonder why the word *complete* is used to describe this property. One might think of it this way. If an axiom system is *complete*, then it is like a complete set of specifications for a corresponding model. All models for the axiom system are essentially the same: they are isomorphic. If an axiom system is *not complete*, then one does *not* have a complete set of specifications for a corresponding model. The specifications are insufficient, some details are not nailed down. As a result, there can be models that differ from each other: models that are not isomorphic.

For an example, consider the following new Axiom System #6.

Axiom System:	Axiom System #6
Primitive Terms:	*ake*, *bem*

Primitive Relations:	relation from *A* the set of all *akes* to *B* the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to.

You'll recognize that Axiom System #6 is just Axiom System #2 without the third axiom.

Recall that in the previous Section 1.2.2, we discussed two interpretations of Axiom System #2: Bob's interpretation and Dan's interpretation. In both of those interpretations, the statements of axiom <1> and <2> were true. But we observed that in Bob's interpretation, the statement of axiom <3> was true, while in Dan's interpretation, the statement of axiom <3> was false. This demonstrated that axiom <3> of Axiom System #2 is an independent axiom. Notice that it also demonstrated that Bob's interpretation *is a model* of Axiom System #2, while Dan's interpretation *is not a model* of Axiom system #2.

Now consider Bob's and Dan's interpretations as being interpretations of Axiom System #6. Observe that for both interpretations, the statements of axioms <1> and <2> are true. From this we conclude that both Bob's and Dan's interpretations *are models* of Axiom System #6.

Now observe that these two models of Axiom System #6 are not isomorphic. (There are others as well.) To see why, note that in Bob's model, there are six segments, but in Dan's model, there is only one segment. To prove that two models are isomorphic, one must demonstrate a one-to-one correspondence between the objects of one model and the objects of the other model. (And one must demonstrate some other stuff, as well.) It would be impossible to come up with a one-to-one correspondence between the objects of Bob's model and the objects of Dan's model, because the two models do not have the same number of objects!

The discussion of Bob's and Dan's models for axiom system #6 can be generalized to the extent that it is possible to formulate an alternate wording of the definition of a complete axiom system. Remember that axiom system #6 has two axioms. Consider a feature that distinguished Bob's model from Dan's model. One obvious feature is the number of segments. As observed above, in Bob's model, there are six segments, but in Dan's model, there is only one segment. Now consider the following statement:

Statement *S*: There are exactly six *bems*.

This Statement *S* is an *additional independent statement* for axiom system #6. By that, I mean that it is *not* one of the axioms for Axiom System #6, and there is a model for axiom system #6 in which Statement *S* is true (Bob's model) and there is a model for axiom system #6 in which Statement *S* is false (Dan's model). The fact that an additonal independent statement can be written regarding the number of line segments indicates that axiom system #6 does not sufficiently specify the number of line segments. That is, axiom system #6 is incomplete.

More generally, if it is possible to write an additional independent statement regarding the primitive terms and relations in an axiom system, then the axiom system is not complete (and vice-versa). Thus, an alternate way of wording the definition of a complete axiom system is as follows:

Definition 8 Alternate definition of a complete axiom system

An axiom system is said to be *not complete* if it *is possible* to write an *additonal independent statement* regarding the primitive terms and relations. (An *additional independent statement* is a statement *S* that is *not* one of the axioms and such that there is a model for the axiom system in which Statement *S* is true and there is also a model for the axiom system in which Statement *S* is false.) An axiom system is said to be *complete* if it *is not possible* to write such an additional independent statement.

We have discussed the fact that Statement *S* "there are exactly six *bems*" is an additional independent statement for Axiom System #6, because the statement is not an axiom and it cannot be proven true on the basis of the axioms. If we wanted to, we could construct a new axiom system #7 by appending an axiom to Axiom System #6 in the following manner.

Axiom System:	Axiom System #7 (Axiom System #6 with an added axiom)
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from *A* the set of all *akes* to *B* the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted *ake*$_1$, *ake*$_2$, *ake*$_3$, *ake*$_4$. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> There are exactly six *bems*.

Of course, Bob's successful interpretation for Axiom System #6 would be a successful interpretation for Axiom System #7, as well. That is, Bob's interpretation is a model for Axiom System #6 and also for Axiom System #7. But Dan's successful interpretation for Axiom System #6 would not be a successful interpretation for Axiom System #7. That is, Dan's interpretation is a model for Axiom System #6, but not for Axiom System #7.

Keep in mind, that we could have appended a different axiom to Axiom System #6.

Axiom System:	Axiom System #8 (Axiom System #6 with a different axiom added)
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from *A* the set of all *akes* to *B* the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted *ake*$_1$, *ake*$_2$, *ake*$_3$, *ake*$_4$. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> There is exactly one *bem*.

We see that Dan's interpretation is a model for Axiom Systems #6 and #8. On the other hand, Bob's interpretation is a model for Axiom System #6 but not for Axiom System #8.

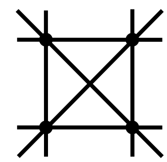
Before going on to read the next subsection, you should do the exercises for the current subsection. The exercises are found in Section 1.4 on page 29.

1.3.2. Don't Use Models to Prove Theorems

Recall Axiom System #6, presented in the previous Section 1.3.1 on page 26. Now consider the following "theorem" and "proof".

Theorem: In Axiom System #6, there are exactly six bems.

Proof:
(1) Here is a model of Axiom System #6. We see that there are exactly six lines.
End of Proof



The proof seems reasonable enough, doesn't it? But wait a minute. That picture above is just one model of Axiom System #6. It is Bob's model. Remember that we also studied Dan's Model of Axiom System #6:  In Dan's model, there is only one line. So the statement of the theorem is not even a true statement for Axiom System #6.

There are two lessons to be learned here:
(1) Don't use a model of an axiom system to prove a statement about an axiom system. It is possible that the statement is true in the particular model that you have in mind, but is not true in general for the axiom system.
(2) Even if you know that a statement about an axiom system is in fact a valid theorem, you still cannot use a particular model to prove the statement. You have to use the axioms.

1.4. Exercises for Chapter 1

Exercises for Section 1.1 Introduction to Axiom Systems

The first two exercises are about Axiom System #1. That axiom system was introduced in Section 1.1.5 and has an undefined relation.

[1] Which of the following interpretations of Axiom System #1 is successful? That is, which of these interpretations is a model of Axiom System #1? Explain.
(a) Interpret the words "x is related to y" to mean "$xy > 0$".
(b) Interpret the words "x is related to y" to mean "$xy \neq 0$".
(c) Interpret the words "x is related to y" to mean "x and y are both even or are both odd".
Hint: One of the three is unsuccessful. The other two are successful. That is, they are models.

[2] Consider the two models of Axiom System #1 that you found in exercise [1]. For each model, determine whether the statement "*1 is related to -1*" is true or false.

[3] (This exercise is about Axiom System #2. That axiom system was introduced in Section 1.1.6 and has undefined terms and an undefined relation.)
Prove Theorem #2 for Axiom System #2.

Theorem #2 for Axiom System #2: For every *ake*, there are exactly three *bems* that the *ake* is related to.

Hint: In Part 1 of your proof, show that there must be at least three *bems*. In Part 2 of your proof, show that there cannot be more than three *bems*. Also be sure to review the **Digression to Discuss Proof Structure** at the end of Section 1.1.6. It specifically mentions the structure that you will need for your proof.

Exercises for Section 1.2.1 Consistency

Exercises [4], [5] and [6] are about Axiom System #1. That axiom system was introduced in Section 1.1.5 and has an undefined relation.)

[4] Is Axiom System #1 consistent? (Hint: Consider your answer to exercise [1].)

[5] Make up an example of a consistent axiom system that includes the axioms of Axiom System #1 plus one more axiom.

[6] Make up an example of an inconsistent axiom system that includes the axioms of Axiom System #1 plus one more axiom.

[7] (This exercise is about Axiom System #4. That axiom system was introduced in Section 1.2.1) Prove that Axiom System #4 is consistent by demonstrating a model. (Hint: Produce a successful interpretation involving a picture of dots and segments.)

Exercises for Section 1.2.2 Independence

Exercises [8] - [11] explore Axiom System #2, which was introduced in Section 1.1.6 on page 15.

[8] The goal is to prove that in Axiom System #2, Axiom <1> is independent. To do this, you must do two things:
- (1) Produce an interpretation for Axiom System #2 in which the statements of Axioms <2> and <3> are true and the statement of Axiom <1> is true. (That is, the statements of all three axioms are true.)
- (2) Produce an interpretation for Axiom System #2 in which the statements of Axioms <2> and <3> are true and the statement of Axiom <1> is false.

[9] The goal is to prove that in Axiom System #2, Axiom <2> is independent. To do this, you must do two things:
- (1) Produce an interpretation for Axiom System #2 in which the statements of all three axioms are true. (Hey, you already did this in question [8]!)
- (2) Produce an interpretation for Axiom System #2 in which the statements of Axioms <1> and <3> are true and the statement of Axiom <2> is false.

[10] The goal is to prove that in Axiom System #2, Axiom <3> is independent. To do this, you must do two things:

(1) Produce an interpretation for Axiom System #2 in which the statements of all three axioms are true. (Hey, you already did this in question [8]!)
(2) Produce an interpretation for Axiom System #2 in which the statements of Axioms <1> and <2> are true and the statement of Axiom <3> is false.

[11] Is axiom system #2 independent? Explain.

Exercises for Section 1.3.1 Completeness

Exercises [12], [13], [14] are about Axiom System #1. That axiom system was introduced in Section 1.1.5 and has an undefined relation.

[12] For Axiom System #1, is the statement "*1 is related to -1*" an independent statement? Explain. (Hint: Consider your answer to exercise [2].)

[13] Based on your answer to exercise [12], is Axiom System #1 complete? Explain.

[14] Make up an example of a statement involving the terms and relations of Axiom System #1 such that the statement is not independent.

Review Exercise for Chapter 1 Axiom Systems

The Review Exercises for Chapter 1 explore the following new Axiom System #9.

Axiom System:	Axiom System #9
Primitive Terms:	*ake*, *bem*
Primitive Relations:	relation from A the set of all *akes* to B the set of all *bems*, spoken "*The ake is related to the bem*".
Axioms:	<1> There are four *akes*. These may be denoted ake_1, ake_2, ake_3, ake_4. <2> For any two distinct *akes*, there is exactly one *bem* that both *akes* are related to. <3> For any *bem*, there are at least two *akes* that are related to the *bem*. <4> For any *bem*, there is at least one *ake* that is not related to the *bem*.

[15] Produce a model for Axiom System #9 that that uses *dots* and *segments* to correspond to *akes* and *bems*, and that uses the words *"the dot touches the segment"* to correspond to the words *"the ake is related to the bem"*. That is, draw a picture that works. (Hint: See if one of the models for Axiom System #2 will work.)

[16] Is Axiom System #9 consistent? Explain.

[17] Again using dots and segments to correspond to *akes* and *bems*, and again using the words *"the dot touches the segment"* to correspond to the words *"the ake is related to the bem"*, produce a model for Axiom System #9 that is not isomorphic to the model that you produced in exercise [15]. Hint: start by drawing one line segment that has 3 dots touching it. Then add to your drawing whatever dots and segments are necessary to make the drawing satisfy the axioms.

[18] Consider the two models of Axiom System #9 that you found in exercises [15] and [17], and consider the statement *S*: “There exist exactly six bems.” Statement S is a statement about the undefined terms of Axiom System #9.

(a) Is Statement *S* true or false for the model that you found in exercise [15]?
(b) Is Statement *S* true or false for the model that you found in exercise [17]?
(c) Is Statement *S* an independent statement for Axiom System #9? (Hint: Consider your answers to parts (a) and (b).)
(d) Is Axiom System #9 complete? Explain. (Hint: Consider your answer to part (c).)

[19] The goal is to prove that in Axiom System #9, Axiom <1> is independent. In question [15], you produced an interpretation involving dots and segments in which the statements of all four axioms are true. (That is, you produced an interpretation that is a model.) Therefore, all you need to do is produce an interpretation for which the statements of Axioms <2>, <3>, and <4> are true and but the statement of Axiom <1> is false. Use dots and segments.

[20] The goal is to prove that in Axiom System #9, Axiom <2> is independent. In question [15], you produced an interpretation involving dots and segments in which the statements of all four axioms are true. (That is, you produced an interpretation that is a model.) Therefore, all you need to do is produce an interpretation for which the statements of Axioms <1>, <3>, and <4> are true and but the statement of Axiom <2> is false. Use dots and segments.

[21] The goal is to prove that in Axiom System #9, Axiom <3> is independent. In question [15], you produced an interpretation involving dots and segments in which the statements of all four axioms are true. (That is, you produced an interpretation that is a model.) Therefore, all you need to do is produce an interpretation for which the statements of Axioms <1>, <2>, and <4> are true and but the statement of Axiom <3> is false. Use dots and segments.

[22] The goal is to prove that in Axiom System #9, Axiom <4> is independent. In question [15], you produced an interpretation involving dots and segments in which the statements of all four axioms are true. (That is, you produced an interpretation that is a model.) Therefore, all you need to do is produce an interpretation for which the statements of Axioms <1>, <2>, and <3> are true and but the statement of Axiom <4> is false. Use dots and segments.

[23] Is axiom system #9 independent? Explain.

2.Axiomatic Geometries

2.1. Introduction and Basic Examples

For the remainder of the course, we will be studying axiomatic geometry. Before starting that study, we should be sure and understand the difference between analytic geometry and axiomatic geometry.

2.1.1. What is an analytic geometry?

Very roughly speaking, an analytic geometry consists of two things:

- a set of points that is represented in some way by real numbers
- a means of measuring the distance between two points

For example, in plane Euclidean analytic geometry, a point is represented by a pair $(x, y) \in \mathbb{R}^2$. That is, a point is an ordered pair of real numbers. The distance between points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is obtained by the fomula $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In three dimensional Euclidean analytic geometry, one adds a z coordinate.

In analytic geometry, objects are described as sets of points that satisfy certain equations. A line is the set of all points (x, y) that satisfy an equation of the form $ax + by = c$; a circle is the set of all points that satisfy an equation of the form $(x - h)^2 + (y - k)^2 = r^2$, etc. Every aspect of the behavior of analytic geometric objects is completely dictated by rules about solutions of equations. For example, any two lines either don't intersect, or they intersect exactly once, or they are the same line; there are no other possibilities. That this is true is simply a fact about simultaneous solutions of a pair of linear equations in two variables:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

You will see that in axiomatic geometry, objects are defined in a very different way, and their behavior is governed in a very different manner.

2.1.2. What is an axiomatic geometry?

Very roughly speaking, an axiomatic geometry is an axiom system with the following primitive (undefined) things.

Primitive Objects:	*point*, *line*
Primitive Relation:	relation from the set of all *points* to the set of all *lines*, spoken *the point lies on the line*

Remark: It is not a very confident definition that begins with the words "…*roughly speaking*…". But in fact, one will not find general agreement about what constitutes an axiomatic geometry. My description above will work for our purposes.

You'll notice, of course, that the axiom system above is essentially the same sort of axiom system that we discussed in Chapter 1. The only difference is that we stick to the particular convention of using undefined objects called *point* and *line*, and an undefined relation spoken *the point is on the line*. It is natural to wonder why we bothered with the meaningless terms *ake* and *bem*, when we could have used the more helpful terms *point* and *line*. The reason for starting with the meaningless terms was to stress the idea that the primitive terms are *always* meaningless; they are not supposed to be helpful. When studying axiomatic geometry, it will be very important to keep in mind that even though you may think that you know what a *point* and a *line* are, you really don't. The words are as meaningless as *ake* and *bem*. On the other hand, when studying a *model* of an axiomatic geometry, we *will* know the meaning of the objects and relations, but we will be careful to always give those objects and relations names other than *point* and *line*. For instance, we used the names *dot* and *segment* in our models that involved drawings. The word *dot* refers to an actual drawn spot on the page or chalkboard; it will be our interpretation of the word *point*, which is an undefined term.

Because the objects and relations in axiomatic geometry are undefined things, their behavior will be undefined as well, unless we somehow dictate that behavior. That is the role of the axioms. Every aspect of the behavior of axiomatic geometric objects must be dictated by the axioms. For example, if we want lines to have the property that two lines either don't intersect, or they intersect exactly once, or they are the same line, then that will have to be specified in the axioms. We will return to the notion of what makes axiomatic points and lines behave the way we "normally" expect points and lines to behave in Section 2.3, when we study *incidence geometry*..

2.1.3. A Finite Geometry with Four Points

A finite axiomatic geometry is one that has a finite number of points. Our first example has four points.

Axiom System:	Four-Point Geometry
Primitive Objects:	*point*, *line*
Primitive Relations:	relation from the set of all *points* to the set of all *lines*, spoken "*The point lies on the line*".
Axioms:	<1> There are four *points*. These may be denoted P_1, P_2, P_3, P_4. <2> For any two distinct *points*, there is exactly one *line* that both *points* lie on. <3> For any line, there exist exactly two points that lie on the line.

Notice that the Four-Point Geometry is the same as Axiom System #2, presented in Section 1.1.6. The differences are minor, just choices of names. In Axiom System #2, the primitive terms are *ake* and *bem*; in the Four-Point Geometry, the primitive terms are *point* and *line*. In Axiom System #2, the primitive relation is spoken "*the ake is related to the bem*"; In the Four-Point Geometry, the primitive relation is spoken "*the point lies on the line*". Following are two theorems of Four-Point Geometry.

Four-Point Geometry Theorem #1: There are exactly six lines.
You will prove this Theorem in the exercises.

Four-Point Geometry Theorem #2: For every point, there are exactly three lines that the point lies on.
You will prove this Theorem in the exercises.

2.1.4. Terminology: Defined Relations

In Chapter 1, you learned that axiom systems can include primitive—that is, undefined—objects and relations. Typically, there will also be objects and relations that are defined in terms of the primitive objects and relations. Here are five new definitions of relations and properties. Each is defined in terms of the primitive objects, primitive relations, and previous definitions.

Definition 9 passes through

- words: Line L *passes through* point P.
- meaning: Point P lies on line L.

The definition above is introduced simply so that sentences about points and lines don't have to always sound the same. For example, we now have two different ways to express Four-Point Geometry Axiom <2>:

- Original wording: For any two distinct *points*, there is exactly one *line* that both *points* lie on.
- Alternate wording: For any two distinct *points*, there is exactly one *line* that passes through both.

The next two definitions are self-explanatory.

Definition 10 intersecting lines

- words: Line L *intersects* line M.
- meaning: There exists a point (at least one point) that lies on both lines.

Definition 11 parallel lines

- words: Line L is parallel to line M.
- symbol: $L||M$.
- meaning: Line L does not intersect line M. That is, there is no point that lies on both lines.

It is important to notice what the definition of parallel lines does not say. It does not say that parallel lines are lines that have the same slope. That's good, because we don't have a notion of slope for line in axiomatic geometry, at least not yet. (It would be in analytic geometry, not axiomatic geometry, that one might define parallel lines to be lines that have the same slope.)

It's also worth noting that we have essentially introduced three new relations. The words "*Line L passes through point P*" indicate a relation from the set of all lines to the set of all points. The words "*Line L intersects line M*" indicate a relation on the set of all lines. Similarly, the words "*Line L is parallel to line M*" indicate another relation on the set of all lines. These relations are not primitive relations, because they have an actual meaning. Those meanings are given by the above definitions, and those definitions refer to primitive terms and relations and to previously defined words. This kind of relation is a *defined relation*.

Here are two additional definitions, also straightforward.

Definition 12 collinear points

- words: The set of points $\{P_1, P_2, \dots, P_k\}$ is collinear.

- meaning: There exists a line L that passes through all the points.

Definition 13 concurrent lines

- words: The set of lines $\{L_1, L_2, \ldots, L_k\}$ is concurrent.
- meaning: There exists a point P that lies on all the lines.

It's worth noting that these two definitions are not relations as we have been discussing relations so far. Rather, they are simply statements that may or may not be true for a particular set of points or a particular set of lines. That is, they are properties that a set of points or a set of lines may or may not have.

It is customary to not list the definitions when presenting the axiom system. This is understandable, because often there are many definitions, and to list them all would be very cumbersome. But it is unfortunate that the definitions are not listed, because it makes them seem less important than the other components of an axiom system, and because the reader often does not remember the definitions and must go looking for them.

2.1.5. Recurring Questions about Parallels

In our study of axiomatic geometry, we will often be interested in the following two questions:

(1) Do parallel lines exist?
(2) Given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L?

The questions are first raised in the current chapter, in discussions of finite geometries. But they will come up throughout the course.

But before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 2.5 on page 56.

2.2. Fano's Geometry and Young's Geometry

2.2.1. Fano's Axiom Sytem and Six Theorems

Let's return to finite geometries. A more complicated finite geometry is the following.

Axiom System:	Fano's Geometry
Primitive Objects:	*point*, *line*
Primitive Relations:	*The point lies on the line.*
Axioms:	<F1> There exists at least one *line*. <F2> For every line, there exist exactly three points that lie on the line. <F3> For every *line*, there exists a *point* that does not lie on the *line*. (at least one *point*) <F4> For any two *points*, there is exactly one *line* that both *points lie on*. <F5> For any two *lines*, there exists a *point* that *lies on* both *lines*. (at least one *point*)

We will study the following six theorems of Fano's Geometry:

Fano's Geometry Theorem #1: There exists at least one point.
Fano's Geometry Theorem #2: For any two lines, there is exactly one point that lies on both lines.
Fano's Geometry Theorem #3: There exist exactly seven points.
Fano's Geometry Theorem #4: Every point lies on exactly three lines.
Fano's Geometry Theorem #5: There does not exist a point that lies on all the lines.
Fano's Geometry Theorem #6: There exist exactly seven lines.

The proofs of the first two theorems of Fano's Geometry are very basic, and are assigned to you in the exercises at the end of this chapter.

In the upcoming subsections 2.2.2 and 2.2.3 and 2.2.4, we will study proofs of Fano's Geometry Theorems #3, #4, and #5. But the proof of Fano's Theorem #6 will wait until a later subsection 2.4.2 (Fano's Sixth Theorem and Self-Duality, which start on page 53).

2.2.2. Proof of Fano's Geometry Theorem #3

The third theorem of Fano's Geometry is easy to state but somewhat tricky to prove.

Fano's Geometry Theorem #3: There exist exactly seven points.

Remark about Proof Structure: Notice that Fano's Theorem #3 is an *existential statement*: It states that something exists. Now consider the five axioms of Fano's Geometry. Notice that axioms <F2>, <F3>, <F4>, and <F5> say something about objects existing, but only in situations where some other prerequisite objects are already known to exist. Those axioms are of no use to us until after we have proven that those other prerequisite objects do exist. Only axiom <F1> says simply that something exists, with no prerequisites. So the proof of Fano's Theorem #3 must start by using axiom <F1>.

Here is a proof of Fano's Theorem #3, with no justifications. In a class drill, you will be asked to provide justifications.

Proof of Fano's Theorem #3

Part 1: Show that there must be at least seven points.
Introduce Line L_1 and points A, B, C, D.

(1) There exists a line. **(Justify.)** We can call it L_1. **(Make a drawing.)**
(2) There are exactly three points on L_1. **(Justify.)** We can call them A, B, C. **(Make a new drawing.)**
(3) There must be a point that does not lie on L_1. **(Justify.)** We can call it D. **(Make a new drawing.)**

Introduce Line L_2 and point E.

(4) There must be a line that both A and D lie on. **(Justify.)**
(5) The line that both A and D lie on cannot be L_1. **(Justify.)** So it must be a new line. We can call it L_2. **(Make a new drawing.)**
(6) There must be a third point that lies on L_2. **(Justify.)**
(7) The third point on L_2 cannot be B or C. **(Justify.)** So it must be a new point. We can call it E. **(Make a new drawing.)**

Introduce Line L_3 and point F.

(8) There must be a line that both B and D lie on. **(Justify.)**

(9) The line that both B and D lie on cannot be L_1 or L_2. **(Justify.)** So it must be a new line. We can call it L_3. **(Make a new drawing.)**

(10) There must be a third point that lies on L_3. **(Justify.)**

(11) The third point on L_3 cannot be A, C, or E. **(Justify.)** So it must be a new point. We can call it F. **(Make a new drawing.)**

Introduce Line L_4 and point G.

(12) There must be a line that both C and D lie on. **(Justify.)**

(13) The line that both C and D lie on cannot be L_1 or L_2 or L_3. **(Justify.)** So it must be a new line. We can call it L_4. **(Make a new drawing.)**

(14) There must be a third point that lies on L_4. **(Justify.)**

(15) The third point on L_4 cannot be A, B, E, or F. **(Justify.)** So it must be a new point. We can call it G. **(Make a new drawing.)**

Part 2: Show that there cannot be an eighth point. (Indirect Proof using the Method of Contradiction)

(16) Suppose there is an eighth point. **(Justify.)** Call it H.

(17) There must be a line that both A and H lie on. **(Justify.)**

(18) The line that both A and H lie on cannot be L_1 or L_2 or L_3 or L_4. **(Justify.)** So it must be a new line. We can call it L_5.

(19) There must be a third point that lies on L_5. **(Justify.)**

(20) Line L_5 must intersect each of the lines L_1 and L_2 and L_3 and L_4. **(Justify.)**

(21) The third point on L_5 must be D. **(Justify. Be sure to explain clearly)**

(22) So points A, D, H lie on L_5.

(23) We have reached a contradiction. **(explain the contradiction)** Therefore, our assumption in step (16) was wrong. There cannot be an eighth point.

End of proof

2.2.3. Proof of Fano's Geometry Theorem #4

The fourth theorem of Fano's geometry is easy to state:

Fano's Geometry Theorem #4: Every point lies on exactly three lines.

The proof is somewhat easier than the proof of the third theorem. The proof is given below without justifications. In the exercises, you will be asked to provide justifications.

Remark About Proof Structure: Note that Fano's Theorem #4 starts with the words "Every point ...". So the theorem is a *universal statement*. It is important to keep in mind that the universal statement of Theorem #4 does not claim that any points exist. It only makes a claim about a point that is already known to exist: a point that is *given*, in other words. So a proof of Theorem #4 must start with a sentence introducing a *generic, given* point. The start of the proof would look something like this:

Start of proof of Fano's Theorem #4

(1) Suppose that a point is given.

(2)

Let me reiterate that one does not begin the proof of Fano's Theorem #4 by proving that a point exists, because the statement of Theorem #4 does not claim that any points exist. Theorem #4 only makes a claim about a *given point*, and so the proof of Theorem #4 must start with the introduction of a *given point*.

(And remember that more generally, in the proof of a universal statement, one starts the proof by stating that a *generic object* is given. A *generic object* is an object or objects that have only the properties mentioned right after the words "for all" in the theorem statement. The given objects are known to have those properties, but are not known to have any other properties. The object of the proof is to prove that the given objects do in fact have some other properties as well.)

Here, then, is the proof of Fano's Theorem #4.

Proof of Fano's Theorem #4

(1) Suppose that P is a point in Fano's geometry.

Part 1: Show that there must be at least three lines that the given point lies on

Introduce Line L_1.

(2) There exist exactly seven points in Fano's geometry. **(Justify.) S**o we can call the given point P_1. There are six remaining points.

(3) Choose one of the six remaining points. Call it P_2. There are five remaining points.

(4) There must be a line that both P_1 and P_2 lie on. **(Justify.)** We can call it L_1.

(5) There must be a third point on L_1. **(Justify.)** Call the third point P_3. So points P_1, P_2, P_3 lie on line L_1. There are four remaining points.

Introduce Line L_2.

(6) Pick one of the four remaining points. Call it P_4. There are now three remaining points.

(7) There must be a line that both P_1 and P_4 lie on. **(Justify.)**

(8) The line that both P_1 and P_4 lie on cannot be L_1. **(Justify.)** So it must be a new line. Call it L_2.

(9) There must be a third point that lies on L_2. **(Justify.)**

(10) The third point on L_2 cannot be P_2 or P_3. **(Justify.)** So it must be one of the three remaining points. Call the third point P_5. So points P_1, P_4, P_5 lie on line L_2. There are two remaining points.

Introduce Line L_3.

(11) Pick one of the two remaining points. Call it P_6. There is now one remaining point.

(12) There must be a line that both P_1 and P_6 lie on. **(Justify.)**

(13) The line that both P_1 and P_6 lie on cannot be L_1 or L_2. **(Justify.)** So it must be a new line. Call it L_3.

(14) There must be a third point that lies on L_3. **(Justify.)**

(15) The third point on L_3 cannot be P_2 or P_3 or P_4 or P_5. **(Justify.)** So it must be the last remaining point. Call the third point P_7. So points P_1, P_6, P_7 lie on line L_3.

Part 2: Show that there cannot be an a fourth line that the given point lies on.

(16) Suppose there is a fourth line that the given point P_1 lies on. **(Justify.)** Call it L_4.

(17) There must be another point on line L_4. **(Justify.)** Call it Q.

(18) Point Q must be either P_2 or P_3 or P_4 or P_5 or P_6 or P_7. **(Justify.)**

(19) Points P_1 and Q both lie on L_4, and they also both lie on one of the lines L_1 or L_2 or L_3.

(20) We have reached a contradiction. **(explain the contradiction)** Therefore, our assumption in step (16) was wrong. There cannot be a fourth line.

End of proof

2.2.4. Proof of Fano's Geometry Theorem #5

Here is the fifth theorem for Fano's Geometry, and a proof with justifications included.

Fano's Geometry Theorem #5: There does not exist a point that lies on all the lines.

Proof of Fano's Theorem #5 (Indirect Proof using the Method of Contradiction)

(1) Suppose that there does exist a point that lies on all of the lines of the geometry. Call it P_1. (assumption)
(2) There exist exactly six other points. (by Theorem #3)
(3) The given point P_1 lies on exactly three lines (by Theorem #4)
(4) Using the notation from the above proof of Theorem #4, we can label the six other points and the three lines as follows:
Points P_1, P_2, P_3 lie on line L_1.
Points P_1, P_4, P_5 lie on line L_2.
Points P_1, P_6, P_7 lie on line L_3.
These are the only lines in the geometry (by step (3) and by our assumption in step (1))
(5) Point P_2 lies on only one line. (by step (4))
(6) Statement (5) contradicts Theorem #4. So our assumption in step (1) was wrong. It cannot be true that there exists a point that lies on all of the lines of the geometry.

End of proof

2.2.5. Models for Fano's Geometry

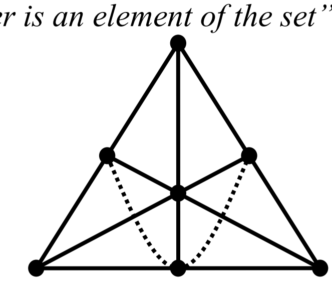
Here are two successful interpretations of Fano's Geometry. That is, here are two models.

Letters and Sets Model of Fano's Geometry

- Interpret points to be the the letters A, B, C, D, E, F, G.
- Interpret lines as sets $\{A,B,C\}$, $\{A,D,E\}$, $\{A,F,G\}$, $\{B,D,F\}$, $\{B,E,G\}$, $\{C,D,G\}$, $\{C,E,F\}$.
- Interpret the words *"the point lies on the line"* to mean *"the letter is an element of the set"*.

Dots and Segments Model of Fano's Geometry.

- Interpret points to be dots in the picture at right.
- Interpret lines to be segments in the picture at right. (The dotted segment is curved.)
- Interpret the words *"the point lies on the line"* to mean *"the dot touches the segment"*.



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2.2.6. Young's Geometry

By changing just the fifth axiom in Fano's Geometry, we obtain Young's Geometry

Axiom System:	Young's Geometry
Primitive Objects:	point, line
Primitive Relations:	relation from the set of all *points* to the set of all *lines*, spoken *the point lies on the line*
Axioms:	<Y1> There exists at least one *line*. <Y2> For every line, there exist exactly three points that lie on the line.

	<Y3> For every *line*, there exists a *point* that does not lie on the *line*. (at least one *point*) <Y4> For any two *points*, there is exactly one *line* that both *points lie on*. <Y5> For each *line L*, and for each *point P* that does not *lie on L*, there exists exactly one *line M* that *passes through P* and is *parallel to L*.

We will not study any theorems of Young's Geometry, but I want you to be aware that it exists and to see a model for it. And even though we won't study any theorems about it, Young's Geometry can help our understanding of Fano's Geometry. Here is a model.

Letters and Sets Model of Young's Geometry

- Interpret points to be the the letters A, B, C, D, E, F, G, H, I.
- Interpret lines to be the sets $\{A, B, C\}$, $\{A, D, G\}$, $\{A, E, H\}$, $\{A, F, I\}$, $\{B, D, H\}$, $\{B, E, I\}$, $\{B, F, G\}$, $\{C, D, I\}$, $\{C, E, G\}$, $\{C, F, H\}$, $\{D, E, F\}$, $\{G, H, I\}$,.

Interpret words *"the point lies on the line*" to mean *"the letter is an element of the set"*.

But before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 2.5 on page 56.

2.3. Incidence Geometry

2.3.1. Incidence Relations and Axioms of Incidence

You will notice that in each of the finite geometries that we have encountered so far, the axioms can be classified into two types.
One type of axiom is just about the primitive objects. Here are two examples.

- Four-Point Geometry Axiom <1>: There exist exactly four points.
- Fano's Axiom <F1>: There exists at least one line.

A second type of axiom is about the behavior of the primitive relation. Here are two examples.

- Four-Point Geometry Axiom <2>. For any two distinct *points*, there is exactly one *line* that both *points* lie on.
- Fano's Axiom <F3>: For every *line*, there exists a *point* that does not lie on the *line*. (at least one *point*)

In some older books on axiomatic geometry, the primitive relation was written "the *line is incident upon* the *point*". Such a primitive relation could be referred to as *the incidence relation*. Axioms such as the two above that described the behavior of the incidence relation were called *axioms of incidence*. Even though most books no longer use the words "*…is incident upon…*" for the primitive relation, it is still fairly common for any axioms that describe the behavior of the primitive relation to be referred to as *axioms of incidence*. This can be confusing.

2.3.2. The Axiom System for Incidence Geometry

To add to the confusion, the following axiom system is usually called *Incidence Geometry*.

Axiom System:	Incidence Geometry
Primitive Objects:	point, line
Primitive Relations:	relation from the set of all *points* to the set of all *lines*, spoken *the point lies on the line*
Axioms:	<I1> There exist two distinct points. (at least two)

	<I2> For every pair of distinct points, there exists exactly one line that both points lie on. <I3> For every line, there exists a point that does not lie on the line. (at least one) <I4> For every line, there exist two points that do lie on the line. (at least two)

We will study the following five theorems of Incidence Geometry:

Incidence Geometry Theorem #1: *In Incidence Geometry, if L and M are distinct lines that intersect, then they intersect in only one point.*
Incidence Geometry Theorem #2: *In Incidence Geometry, there exist three points that are not collinear.*
Incidence Geometry Theorem #3: *In Incidence Geometry, there exist three lines that are not concurrent*
Incidence Geometry Theorem #4: *In Incidence Geometry, for every point P, there exists a line L that does not pass through P.*
Incidence Geometry Theorem #5: *In Incidence Geometry, for every point P, there exist at least two lines that pass through P.*

Notice that the axioms for Incidence Geometry are less specific than any axiom system that we have seen so far. The precise number of points is not even specified. Even so, notice that axioms <I2>, <I3>, and <I4> do guarantee that the primitive points and lines in Incidence Geometry will have some of the "normal" behavior that we associate with points and lines in analytic geometry, or in drawings that we have made since grade school.

For instance, we are used to the fact that two lines can either be parallel, or intersect once, or be the same line. That is, distinct lines that are not parallel can only intersect once. It was mentioned in Section 2.1.1 that in analytic geometry, lines behave this way as a consequence of behavior of solutions of systems of linear equations. In Section 2.1.2, it was mentioned that in axiomatic geometry, every aspect of the behavior of points and lines will need to be specified by the axioms. The axioms of Incidence Geometry do in fact guarantee that lines have the particular behavior we are discussing. Notice that Incidence Geometry Theorem #1 articulates this fact.

2.3.3. Proof of Incidence Geometry Theorem #1

Observe that Theorem #1 is a conditional statement.

Incidence Geometry Theorem #1: *In Incidence Geometry, if L and M are distinct lines that intersect, then they intersect in only one point.*

The most common structure for the proof of a conditional statement is a sort of frame, with the entire hypothesis of the theorem written in statement (1) of the proof, as a given statement, and with the entire conclusion of the theorem written in the final statement of the proof, with some justification. For Incidence Geometry Theorem #1, such a frame would look like this:

Proof
(1) In Incidence Geometry, suppose that L and M are distinct lines that intersect. (given)

*
* (some steps here)
*

(*) Conclude that lines L and M only intersect in one point. (some justification)
End of Proof

The steps in the middle must somehow bridge the gap between the first, given, statement and the final statement.

Here is a proof of Incidence Geometry Theorem #1 with all of the steps shown but without justificiations. (Notice the frame, the proof structure.) You will be asked to provide the justifications and identify the contradicton in an exercise.

Proof of Incidence Geometry Theorem 1.

(1) In Incidence Geometry, suppose that L and M are distinct lines that intersect. **(Justify.)**
(2) Since lines L and M intersect, there must be at least one point that both lines L and M pass through (by definition of *intersect*). We can call one such point P.
(3) Assume that there is more than one point that that both lines pass through. **(Justify.)** Then there is a second point, that we can call Q.
(4) Observe that there are two lines that pass through points P and Q.
(5) We have reached a contradiction. **(Identify the contradiction.)** So our assumption in (3) was wrong. There cannot be more than one point that that both lines pass through.
(6) Conclude that lines L and M only intersect in one point. (by (1), (2), and (5))

End of Proof

2.3.4. Proof of Incidence Geometry Theorem #2

Observe that Theorem #2 is an existential statement.

Incidence Geometry Theorem #2: *In Incidence Geometry, there exist three points that are not collinear.*

Now consider the four axioms of Incidence Geometry. Notice that axioms <I2>, <I3>, and <I4> say something about objects existing, but only in situations where some other prerequisite objects are already known to exist. Those axioms cannot be used until it has been proven that those other prerequisite objects do exist. Also notice that Incidence Geometry Theorem #1 does not give us the existence of any objects. (Thereom #1 only tells us something about the situation where we already know that two distinct, intersecting lines exist.) Only axiom <I1> says simply that something exists, with no prerequisites. So the proof of Incidence Geometry Theorem #2 must start by using axiom <I1>.

Here is a proof of the theorem with all of the steps shown but without justificiations. You will be asked to provide the justifications in an exercise.

Proof of Incidence Geometry Theorem #2.

(1) In Incidence Geometry, two distinct points exist. **(Justify.)** Call them P and Q.
(2) A line exists that passes through P and Q. **(Justify.)** Call the line L.
(3) There exists a point that does not lie on L. **(Justify.)** Call the point R.

(4) We already know (by statement (3)) line L does not pass through all three points P, Q, R. But suppose that some other line M does pass through all three points. (assumption)
(5) Observe that points P and Q both lie on line L and also both lie on line M.
(6) Statement (5) contradicts something. **(Explain the contradiction.)** Conclude that our assumption in step (4) was wrong. That is, there cannot be a line M that passes through all three points P, Q, R. Conclude that the points P, Q, R are non-collinear.

End of Proof

2.3.5. Proof of Incidence Geometry Theorem #3

Observe that Theorem #3 is an existential statement.

Incidence Geometry Theorem #3: *In Incidence Geometry, there exist three lines that are not concurrent.*

Keep in mind that to prove this theorem, we can use any of the four Incidence Geometry axioms and also the first two Incidence Geometry Theorems. Of those six statements, only axiom <I1> and Theorem #2 say that some objects exist, with no prerequisites. So our proof of Theorem #3 must start by using either axiom <I1> or Theorem #2.

Here is a proof of the theorem with all of the steps shown but without justificiations. You will be asked to provide the justifications in an exercise.

Proof of Incidence Geometry Theorem #3
Part I: Introduce three lines L, M, N.
(1) There exist three non-collinear points. **(Justify.)** Call them A, B, C.
(2) There exists a unique line that passes through points A and B. **(Justify.)** Call it L.
(3) Line L does not pass through point C. **(Justify.)**
(4) Similarly, there exists a line M that passes through B and C and does not pass through A, and a line N that passes through C and A and does not pass through B.

Part II: Show that lines L, M, N are not concurrent.
(5) Suppose that lines L, M, N are concurrent. That is, suppose that there exists a point that all three lines L, M, N pass through. **(Justify.)**
(6) Any point that all three lines L, M, N pass through cannot be point A, B, or C. **(Justify.)** So the point that all three lines pass through must be a new point that we can call point D.
(7) There are two lines that pass through points A and D. **(Justify.)**
(8) We have reached a contradiction. **(Explain the contradiction.)** So our assumption in (5) was wrong. Lines L, M, N must be non-concurrent.

End of Proof

2.3.6. Proof of Incidence Geometry Theorem #4

Observe that Theorem #4 is a universal statement.

Incidence Geometry Theorem #4: *In Incidence Geometry, for every point P, there exists a line L that does not pass through P.*

As we have discussed before in this book, a direct proof of such a universal statement would have to use the following form, starting with the statement that a generic object is given:

Direct Proof of Incidence Geometry Theorem #4

(1) In Incidence Geometry, suppose that a point P is given.
*
* (Some steps here, including introduction of a line called L, with the existence of L justified somehow by the axioms and prior theorems of Incidence Geometry)
*
(*) Conclude that line L does not pass through P.

End of Proof

But it is possible to write a very simple indirect proof of Theorem #4, using the method of contradiction. In order to do that, we will need to know how to write the negation of the statement of the theorem. Hopefully, you have studied the negation of quantified logical statements in a previous course. Here is a review (or a crash course, if you have not previously studied the topic).

Digression to Discuss the Negation of Quantified Logical Statements

We will call the statement of the theorem Statement *S*.

S: "For every point P, there exists a line L that does not pass through P."

This statement can be abbreviated, using logical symbols:

***S* abbreviated:** "∀ point P, (∃ line L such that (L does not pass through P))."

The negation of statement *S* can formed most easily by simply parking the words "It is not true that..." in front. In symbols, one can simply park the negation symbol, "~", in front.

~S: "It is not true that for every point P, there exists a line L that does not pass through P."
~S abbreviated: "~(∀ point P, (∃ line L such that (L does not pass through P)))."

Notice that although the sentence ~*S* presented above is the negation of *S* and it was easy to create, the form in which it is presented is not very helpful. When dealing with the negations of logical statements, it is generally more helpful to rewrite the statements in a form where the negation symbol appears as far to the right as possible. We will rewrite statement ~S in steps, with the negation symbol moving one notch to the right in each step.

When we move the negation symbol to the right in statement ~*S*, we will have to change the wording of the quantifiers. Here are the rules:

- When the symbol ~ moves to the right across a universal quantifier, the quantifier changes to an existential quantifier. That is, we make the following replacement
  - Replace a statement of this form: "$\sim\forall x$, predicate"
  - with this form:"$\exists x$ such that ~predicate".
- When the symbol ~ moves to the right across an existential quantifier, the quantifier changes to a universal quantifier. That is, we make the following replacement
  - Replace a statement of this form this:"$\sim\exists x$ such that predicate"

- with this: "$\forall x$, ~predicate".

Those instructions are awfully vague, but after a couple of examples, hopefully you will find that the instructions are helpful enough. Let's rewrite statement ~*S* in steps, moving the negation symbol ~ one notch to the right in each step. It is easies to work with the abbreviated form of the statement first. For reference, I will show the original form of the negation first:

The original form of the negation

~S: "It is not true that for every point P, there exists a line L that does not pass through P."
~S abbreviated: "~(∀ point P, (∃ line L such that (L does not pass through P)))."

Step one: move the ~symbol one notch to the right. The result is

~S abbreviated: "∃ point P such that ~(∃ line L such that (L does not pass through P))."
~S: "There exists a point P such that it is not true that there exists a line L such that L does not pass through P."

Step two: move the ~symbol another notch to the right. The result is

~S abbreviated: "∃ point P such that for ∀ line L, ~(L does not pass through P)."
~S: "There exists a point P such that for every line L, it is not true that L does not pass through P."

At this point, the negation symbol ~ is sitting in front of a phrase that is easy to negate. That is, we can simply rewrite the statements as follows.

Step three: rewrite the statement. The result is

~S abbreviated: "∃ point P such that for ∀ line L, L passes through P)."
~S: "There exists a point P such that for every line L, L passes through P."

This version of the negation of ~*S* is much easier to understand than the original.

End of Digression to Discuss the Negation of Quantified Logical Statements

Now that we know how the negation of statement *S* looks, we are ready to write a proof of Incidence Geometry Theorem #4 using the Method of Contradiction.

Proof Incidence Geometry Theorem #4 (Indirect Proof by Method of Contradiction)

(1) In Incidence Geometry, assume that the statement of Theorem #4 is false. That is, suppose that there exists a point P such that for every line L, L passes through P.
(2) The set of all lines is *concurrent*. (by (1) and the definition of concurrent lines)
(3) Statement (2) contradicts Incidence Geometry Theorem #3, which says that there exist three lines that are not concurrent. Therefore our assumption in step (1) was wrong. Conclude that the statement of Theorem #4 is true.

End of Proof

2.3.7. Proof of Incidence Geometry Theorem #5

The first thing to notice about our final Incidence Geometry Theorem is that it is a universal statement.

Incidence Geometry Theorem #5: *In Incidence Geometry, for every point P, there exist at least two lines that pass through P.*

Therefore, the proof must start with the introduction of a generic given point P. That is, the given point P is known to have only the attributes mentioned immediately after the "for every" statement and before the comma. In the statement of Theorem #5, nothing else is said about point P before the comma. That is, the point P is not known to have any attributes at all, other than merely existing. (It will be the goal of the proof to show that P does in fact have some other attirbutes, namely that there exist two lines that pass through it.) So the proof of Theorem #5 will have to have the following form:

Proof of Incidence Geometry Theorem #5

(1) In Incidence Geometry, suppose that a point P is given.
*
* (Some steps here.)
*
(*) Conclude that there exist at least two lines that pass through P.

End of Proof

That settles the outer frame of the proof. The inner steps of the proof will use the method of Proof by Division into Cases. We should review how that works.

Digression to Discuss the Method of Proof by Division into Cases

The Rule of Proof by Division into Cases is presented in lists of Rules of Inference in the following basic form:

Rule of Proof by Division into Cases (simplest form, with two cases)	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$

The statement of the rule is just four lines, but when the rule is used in a proof, the first three lines may correspond to whole sections of the proof. It is important to highlight the appearance of the first three lines in the proof, whether they appear as single lines or as a section. And it is important to make the statement of the fourth line of the rule (the conclusion) very clear.

In the proof of Incidence Geometry Theorem #5, the use of the Rule of Proof by Division into Cases will be spread out. Here is a table that shows how the lines in the rule correspond to the lines or sections in the proof.

Line in the Rule	Corresponding Statements or Sections in the Proof of Theorem #5
$p \vee q$	(3) There are two possibilities: • Either P is one of the three points from statement (2) • or P is not one of the three points from statement (2).

$p \to r$	**Case 1** (4) Suppose that P is one of the three points from statement (2). (5) (6) (7) (8) Therefore there are at least two distinct lines through P in this case.
$q \to r$	**Case 2** (9) Suppose that P is not one of the three points from statement (2). (10) (11) (12) Therefore there are at least two distinct lines through P in this case.
$\therefore r$	**Conclusion** (13) Conclude that in either case, there exist at least two distinct lines through P..

End of Digression to discuss the Method of Proof by Division into Cases

Now that we have discussed the Method of Proof by Division into Cases, the full proof of Theorem #5 should be easier to understand. Here is the theorem and its full proof. You will be asked to justify some of the statements in the proof in a homework exercise.

Incidence Geometry Theorem #5: *In Incidence Geometry, for every point P, there exist at least two lines that pass through P.*

Proof

(1) In Incidence Geometry, suppose that a point P is given.

(2) There exist three non-collinear points. **(Justify.)**

(3) There are two possibilities:

- Either P is one of the three points from statement (2)
- or P is not one of the three points from statement (2)

Case 1

(4) Suppose that P is one of the three points from statement (2). Then let the other two points from statement (2) be named B and C. So the three non-collinear points are P, B, C.

(5) There exists a line through points P and B. **(Justify.)** Call it $\overleftrightarrow{PB}$.

(6) There exists a line through points P and C. **(Justify.)** Call it $\overleftrightarrow{PC}$.

(7) Lines $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$ are not the same line. **(Justify.)**

(8) Therefore there are at least two distinct lines through P in this case

Case 2

(9) Suppose that P is not one of the three points from statement (2). Then let the three points from statement (2) be named A, B, C.

(10) Lines $\overleftrightarrow{PA}, \overleftrightarrow{PB}, \overleftrightarrow{PC}$ exist. **(Justify.)**

(11) Notice that is possible that two of the symbols in statement (10) could in fact represent the same line. For example, even though points P, A, B are distinct points, it could be that they are collinear. If that were the case, then the symbols $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ would represent the same line. But regardless of whether or not P is collinear with certain pairs of the points A, B, C, we know that the three symbols $\overleftrightarrow{PA}, \overleftrightarrow{PB}, \overleftrightarrow{PC}$ cannot all three represent the same line. **(Justify. That is, explain how we know that the symbols $\overleftrightarrow{PA}, \overleftrightarrow{PB}, \overleftrightarrow{PC}$ cannot all**

three represent the same line.) That is, the three symbols $\overleftrightarrow{PA}, \overleftrightarrow{PB}, \overleftrightarrow{PC}$ represent at least two distinct lines, and possibly three.

(12) Therefore there are at least two distinct lines through P in this case

Conclusion

(13) Conclude that in either case, there exist at least two distinct lines through P.

End of Proof

2.3.8. Summaryof Proofs of Incidence Geometry Theorems

In our proofs of the five Incidence Geometry theorems, we discussed a number of basic concepts of proof structure:

- We discussed the simplest "frame" structure for the proof of a conditional statement a statement of the form *If A then B*). (in the proof of Theorem #1)
- We discussed the fact that in the proof of an existence statement, one must start the proof with a statement about something existing, and go from there (in the proofs of Theorems #2 and #3)
- We used the method of proof by contradiction (in the proofs of Theorems #1 and #4), and discussed the importance of knowing how to form the negations of quantified logical statements when proving by method of contradiction (in the proof of Theorem #4)
- We discussed the fact that in the proof of a universal statement, one must start the proof by stating that a generic object is given. (In the discussions of Theorems #4 and #5)
- We used the method of proof by division into cases (in the proof of Theorem #5).

These proof structure concepts are crucial to learning how to read and write proofs. We will revisit them throughout the book.

2.3.9. Models of Incidence Geometry; Abstract versus Concrete Models; Relative versus Absolute Consistency

It is interesting to consider the question of whether or not the Axiom System for Incidence Geometry is Consistent. That is, whether or not it is possible to find a model. Remember that a model is a successful interpretation. Let's try using Four-Point Geometry as an interpretation. (Recall that Four-Point Geometry was introduced in Section 2.1.3 on page 34.

objects in Incidence Geometry	↔	objects in the Four-Point Geometry
points	↔	points
lines	↔	lines
relation in Incidence Geometry	↔	relation in Four-Point Geometry
the point lies on the line	↔	the point lies on the line

To determine whether or not the interpretation is successful, we use the interpretation to translate the axioms of Incidence Geometry into statements in Four-Point Geometry and then consider whether or not the resulting statements are true in Four-Point Geometry.

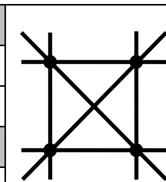
Incidence Geometry axioms	➔	statements in Four-Point Geometry	True?
<I1> There exist two distinct points. (at least two)	➔	Statement 1: There exist two distinct points. (at least two)	true because of Four-Point Geometry Axiom <1>

<I2> For every pair of distinct points, there exists exactly one line that both points lie on.	➔	Statement 2: For every pair of distinct points, there exists exactly one line that both points lie on.	true because of Four-Point Geometry Axiom <2>
<I3> For every line, there exists a point that does not lie on the line. (at least one)	➔	Statement 3: For every line, there exists a point that does not lie on the line. (at least one)	true because of Four Point Geometry Axioms <3> and <1>
<I4> For every line, there exist two points that do lie on the line. (at least two)	➔	Statement 3: For every line, there exist two points that do lie on the line. (at least two)	true because of Four-Point Geometry Axiom <3>

The table above demonstrates that Four-Point Geometry can provide a successful interpretation of Incidence Geometry. That is, Four-Point Geometry can be a model of Incidence Geometry. This demonstrates that Incidence Geometry is consistent. Sort of…

The concept of consistency seemed to be about demonstrating that the words of the axiom system could be interpreted as actual, concrete things. It is a little unsatisfying that we have only demonstrated that the words of Incidence Geometry can be interpreted as other words from another axiom system. That's a little bit like paying back an I.O.U. with another I.O.U. It would be more satisfying if we had an interpretation involving actual, concrete things. Here's one that uses a picture:

objects in Incidence Geometry	↔	objects in the picture at right
points	↔	dots
lines	↔	line segments
relation in Incidence Geometry	↔	relation in the picture at right
the point lies on the line	↔	the dot touches the line segment



To determine whether or not the interpretation is successful, we use the interpretation to translate the axioms of Incidence Geometry into statements about the picture, and then consider whether or not the resulting statements about the picture are true.

axioms of Incidence Geometry	➔	statements about the picture	True?
<I1> There exist two distinct points. (at least two)	➔	Statement 1: There exist two distinct dots. (at least two)	true
<I2> For every pair of distinct points, there exists exactly one line that both points lie on.	➔	Statement 2: For every pair of distinct dots, there exists exactly one line segment that both dots touch.	true
<I3> For every line, there exists a point that does not lie on the line. (at least one)	➔	Statement 3: For every line, segment there exists a dot that does not touch the line segment. (at least one)	true
<I4> For every line, there exist two points that do lie on the line. (at least two)	➔	Statement 4: For every line segment, there exist two dots that touch the line segment. (at least two)	true

The table above demonstrates that the picture with four dots and six line segments can provide a successful interpretation of Incidence Geometry. That is, the picture with four dots and six line segments can be a model of Incidence Geometry. This demonstrates that Incidence Geometry is consistent.

There is terminology that applies to the above discussion

Definition 14 Abstract Model, Concrete Model, Relative Consistency, Absolute Consistency

- An abstract model of an axiom system is a model that is, itself, another axiom system.
- A concrete model of an axiom system is a model that uses actual objects and relations.
- An axiom system is called relatively consistent if an abstract model has been demonstrated.
- An axiom system is called absolutely consistent if a concrete model has been demonstrated.

The Four-Point Geometry is an example of an abstract model for Incidence Geometry. The picture with four dots and six line segments is an example of a concrete model.

It is possible for an axiom system to be both relatively consistent and absolutely consistent. The fact that Four-Point Geometry is a model for Incidence Geometry merely proves that Incidence Geometry is Relatively Consistent. But we can also say that Incidence Geometry is absolutely consistent because of the model involving the picture with four dots.

In the exercises, you will prove that Fano's Geometry and Young's Geometry are also models of incidence geometry. Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 2.5 on page 56.

2.4. Advanced Topic: Duality

2.4.1. The Four-Line Geometry and Duality

Look back at the Four-Point Geometry, presented in Section 2.1.3. Using the find & replace feature in a word processor, we can make the following replacements in the axiom system to create a new axiom system.

- Replace every occurrence of *point* in the original with *line* in the new axiom system.
- Replace every occurrence of *line* in the original with *point* in the new axiom system.
- Replace every occurrence of *lies on* in the original with *passes through* in the new.

The resulting Axiom System is as follows:

Axiom System:	Four-Line Geometry
Primitive Objects:	*line*, *point*
Primitive Relations:	relation from the set of all *lines* to the set of all *points*, spoken "*The line passes through the point*".
Axioms:	<1> There are four *lines*. These may be denoted $line_1$, $line_2$, $line_3$, $line_4$. <2> For any two distinct *lines*, there is exactly one *point* that both *lines* pass through. <3> For any *point*, there are exactly two *lines* that pass through the *point*.

Remember that there were two theorems in Four-Point Geometry.

Four-Point Geometry Theorem #1: *There are exactly six lines.*

Four-Point Geometry Theorem #2: *For every point, there are exactly three lines that the point lies on.*

Realize that we can use the same find & replace operations to change the wording of the statements of those two theorems. We obtain two new statements.

New Statement #1: *There are exactly six points.*

New Statement #2: *For every line, there exist exactly three points that the line passes through.*

Are these new statements true in Four-Line Geometry? That is, can they be proven using a proof that refers to the axioms of Four-Line Geometry? Well, remember that the new statements are translations of statements that are theorems in the Four-Point Geometry. We can use the same find & replace operations to change the wording of the proofs of the Four-Point Theorems, and the result will be new proofs of the corresponding Four-Line Theorems. In other words, the theorems of Four-Point Geometry translate into valid theorems in Four-Line Geometry:

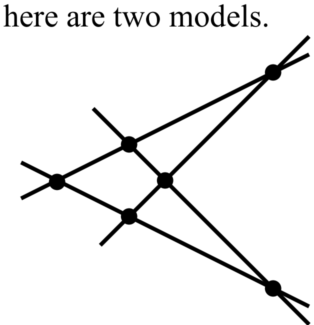
Four-Line Geometry Theorem #1: *There are exactly six points.*

Four-Line Geometry Theorem #2: *For every line, there exist exactly three points that the line passes through.*

Here are two successful interpretations of Four-Line Geometry. That is, here are two models.

Segments and Dots Model of the Four-Line Geometry.

- Interpret lines to be segments in the picture at right.
- Interpret points to be dots in the picture at right.
- Interpret the words *"the line passes through the point*" to mean *"the segment touches the dot"*.



Sets and Letters Model of the Four-Line Geometry

- Interpret lines to be the sets $\{A, B, C\}$, $\{A, D, E\}$, $\{B, E, F\}$, and $\{C, D, F\}$.
- Interpret points to be the the letters A, B, C, D, E, and F.
- Interpret the words *"the line passes through the point*" to mean *"the set contains the letter"*.

What we have just done is very significant: we were able to describe a new axiomatic geometry and state a valid theorems about it without having to do any new work to prove the theorems. The only work involved was that we had to be very clear about the translation that we used. The translation that we used is often useful when studying axiomatic geometry. The underlying concept is called *duality* and is described in the following definition.

Definition 15 the concept of *duality* and the *dual* of an axiomatic geometry

Given any axiomatic geometry with primitive objects *point* and *line*, primitive relation "*the point lies on the line*", and defined relation "*the line passes through the point*", one can obtain a new axiomatic geometry by making the following replacements.

- Replace every occurrence of *point* in the original with *line* in the new axiom system.
- Replace every occurrence of *line* in the original with *point* in the new axiom system.
- Replace every occurrence of *lies on* in the original with *passes through* in the new.
- Replace every occurrence of *passes through* in the original with *lies on* in the new.

The resulting new axiomatic geometry is called the *dual* of the original geometry. The dual geometry will have primitive objects *line* and *point*, primitive relation "*the line passes through the point*", and defined relation "*the point lies on the line*." Any theorem of the original axiom system can be translated as well, and the result will be a valid theorem of the new dual axiom system.

We will use the concept of duality occasionally throughout this course. Note that in general, the dual of an axiomic geometry is different from the original axiomatic geometry. For example, the Four-Point Geometry has four points and six lines, while its dual, the Four-Line Geometry, has four lines and six points. The two geometries are not the same.

2.4.2. Fano's Sixth Theorem and Self-Duality

In the previous section, we presented the Four-Line Geometry as the dual of the Four-Point Geometry. We discussed the fact that the proven theorems of the Four-Point Geometry could be recycled in the sense that they could be translated into theorems of the Four-Line Geometry. Because we had already proved the Four-Point Geometry Theorems, we did not need to prove the Four Line Geometry theorems. Or rather, we could prove them by simply pointing out the dual nature of the two geometries. Duality is a sophisticated concept, and you might wonder if it might have been simpler to just skip the introduction of duality and just prove the Four-Line Geometry theorems from scratch. You would be right about that: the Four-Line Geometry theorems could have been proven with short proofs that did not mention duality.

But the point of duality is that there are situations where one can avoid a very long proof by using duality. In this section, we will apply the principle of duality to prove the sixth theorem for Fano's Geometry. The theorem is very easy to state.

Fano's Geometry Theorem #6: There exist exactly seven lines.

But the proof of Fano's Geometry Theorem #6 is hard. It would be good to approach the proof in a smart way. I will discuss two approaches that could be called the "hedgehog" and the "fox" approaches. The ancient Greek poet Archilochus wrote "…the fox knows many little things, but the hedgehog knows one big thing…". In a famous essay, the 20^{th}-century philosopher Isaiah Berlin expanded upon this idea to divide writers and thinkers into two categories: hedgehogs, who view the world through the lens of a single defining idea, and foxes who draw on a wide variety of experiences and for whom the world cannot be boiled down to a single idea. (Thanks, Wikipedia!)

In math, it is good to be at times a hedgehog and at other times a fox. The first time I saw Fano's Theorem #6, I proved it like a hedgehog, using the same approach that worked in my proofs of Fano's Theorems #3 and #4. Here's an outline of my proof.

Hedgehog's Proof of Fano's Theorem #6: There exist exactly seven lines
Part 1 Show that at least seven lines exist.
Introduce Line L_1.
Introduce Line L_2.
Introduce Line L_3.
Introduce Line L_4.
Introduce Line L_5.
Introduce Line L_6.
Introduce Line L_7.
Part 2 Show that there cannot be an eighth line.

It was somewhat comforting to know that the same approach that worked in proving Fano's Theorem #3 and #4 would work to prove Fano's Theorem #6. But the resulting proof was 48 steps long. Let's think like a fox, and see if we can shorten that proof somehow. Or maybe eliminate the proof altogether…

Recall the concept of *duality*, from Definition 15 on page 52. Consider the dual of Fano's Geometry. Remember that we obtain the dual by doing word substitions according to Definition 15. For clarity, I will number the resulting dual axioms with the prefix DF.

Axiom System:	The dual of Fano's Geometry
Primitive Objects:	*line*, *point*
Primitive Relations:	*The line passes through the point.*
Axioms:	<DF1> There exists at least one *point*. <DF2> Exactly three *lines pass through* each *point*. <DF3> There does not exist a *point* that all the *lines* of the geometry *pass through*. <DF4> For any two *lines*, there is exactly one *point* that both *lines pass through*. <DF5> For any two *points*, there is at least one *line* that *passes through* both *points*.

The same word substitutions give us the following list of theorems in the Dual of Fano's Geometry. (

Dual of Fano's Geometry Theorem #1: There exists at least one line.
Dual of Fano's Geometry Theorem #2: For any two points, there is exactly one line that passes through both points.
Dual of Fano's Geometry Theorem #3: There exist exactly seven lines.
Dual of Fano's Geometry Theorem #4: Every line passes through exactly three points.
Dual of Fano's Geometry Theorem #5: There does not exist a line that passes through all the points of the geometry.

By the principle of Duality, we know that the five theorems are valid theorems in the Dual of Fano's Geometry. Think for a minute what that means. It means that the Dual of Fano's Theorems #1 through #5 can be proven using proofs based on the Dual of Fano's Axioms, that is, statements <DF1> through <DF5>. (We don't have to write down those proofs, because we know that they are simply translations of the proofs of Fano's Theorems #1 through #5.) If we assume that statements <DF1> through <DF5> are true statements (make them axioms) then the Dual of Fano's Theorems #1 through #5 are also true statements.

But what if we were in a situation where statements <DF1> through <DF5> are *known* to be true, so that we didn't have to *assume* that they were true? Well, in that situation, we would know that the Dual of Fano's Theorems #1 through #5 are also true statements, because the same proofs mentioned above would still work.

With that in mind, let's consider the truth of statements <DF1> through <DF5> in Fano's Geometry.

Statement <DF1>: There exists at least one *point*.
This statement is true in Fano's Geometry because of Fano's Geometry Theorem #1: There exists at least one point.

Statement <DF2>: Exactly three *lines pass through* each *point*.
This statement is true in Fano's Geometry because of Fano's Geometry Theorem #4: Every *point lies on* exactly three *lines*.

Statement <DF3>: There does not exist a *point* that all the *lines* of the geometry *pass through*.
This statement is true in Fano's Geometry because of Fano's Geometry Theorem #5: There does not exist a point that lies on all the lines.

Statement <DF4>: For any two *lines*, there is exactly one *point* that both *lines pass through*.
This statement is true in Fano's Geometry because of Fano's Geometry Theorem #2: For any two lines, there is exactly one point that lies on both lines.

Statement <DF5>: For any two *points*, there is at least one *line* that *passes through* both *points*.
This statement is true in Fano's Geometry because of Fano's Geometry Axiom <F4> For any two *points*, there is exactly one *line* that both *points lie on*.

We see that statements <DF1> through <DF5> are true in Fano's Geometry. Therefore, we know that in Fano's Geometry, the Dual of Fano's Theorems #1 through #5 are also true statements. In particular, the Dual of Fano's Theorem #3 is true. This theorem says that there exist exactly seven lines. But that statement is also the claim of Fano's Theorem #6. So we have proven Fano's Theorem #6 without having to write down a new proof of it. That is foxy.

It would be good to summarize what we did in an outline.

Fox's Proof of Fano's Theorem #6: There exist exactly seven lines.

Part 1: Write down the statements of the five axioms for the Dual of Fano's Geometry. These are denoted <DF1>, <DF2>, … , <DF5>.
Part 2: Demonstrate that statements <DF1>, <DF2>, … , <DF5> are actually true in Fano's Geometry, by referring to Fano's Theorems #1, #4, #5, #2, and Fano's Axiom <F4>.
Part 3: Point out that by the principal of duality, the Dual of Fano's Theorems #1 through #5 are automatically true in Fano's Geometry.
Part 4: (Conclusion) Point out that the Dual of Fano's Theorem #3 says that there exist exactly seven lines.

End of Proof

It might seem that Fox's proof is actually longer—its development has taken two full pages—and more conceptually difficult than Hedgehog's proof. But after encountering a few examples that make use of the principle of duality, you will find that it will not seem so difficult, and that it can significantly shorten and clarify proofs.

Note that the key to Fox's proof of Fano's Theorem #6 was the fact that all of the statements of the axioms for the Dual of Fano's Geometry are true statements in Fano's Geometry, itself. There is a name for this: we say that Fano's Geometry is *self-dual*.

Definition 16 self-dual geometry

An axiomatic geometry is said to be *self-dual* if the statements of the dual axioms are true statements in the original geometry.

It is important to remember that most geometries are not self-dual. For example, in 2.1.3, we studied the Four-Point Geometry. Four Point Geometry Theorem #2 says that there are exactly six lines. In Section 2.4.1 on page 51 we studied the dual geometry, called the Four-Line Geometry. Four-Line Geometry Axiom <1> says that there are exactly four lines. This axiom of Four-Line Geometry is not a true statement in Four-Point Geometry. So Four-Point Geometry is not self-dual.

2.5. Exercises For Chapter 2

Exercises for Section 2.1 Introduction and Basic Examples

Exercises [1] - [5] are about Four-Point Geometry, introduced in Section 2.1.3.

In the reading, it was pointed out that Four-Point Geometry is basically just Axiom System #2 from Section 1.1.6, but with some of the wording changed. All proofs of statements about Axiom System #2 can be recycled, with their wording changed, into proofs of corresponding statements about the Four-Point Geometry. In exercises [1] - [3], you will do some of this recycling.

[1] Prove Four-Point Geometry Theorem #1. (Hint: translate the proof of Theorem #1 of Axiom System #2 that is presented in the text.)

[2] Prove Four-Point Geometry Theorem #2. (Hint: translate the proof of Theorem #2 of Axiom System #2 that you produced in Section 1.4 Exercise [3].)

[3] Are the Four-Point Geometry axioms independent? Explain. (Hint: Study Section 1.4 Exercises [8] - [11].)

The remaining two exercises about Four-Point Geometry are not adaptations of facts presented in our discussion of Axiom System #2. They are new.

[4] Prove that in the Four-Point Geometry, parallel lines exist. (In Section 2.1.5, you were introduced to two recurring questions about parallel lines in axiomatic geometry. This is one of the questions.)

[5] In the Four-Point Geometry, given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L? Explain. (This is the other recurring question about parallel lines in axiomatic geometry.)

Exercises for Section 2.2 Fano's Geometry and Young's Geometry
Exercises [6] - [12] are about Fano's Geometry, introduced in Section 2.2.1 on page 36.

[6] Prove Fano's Geometry Theorem #1. (presented in Section 2.2.1, on page 36.)

[7] Prove Fano's Geometry Theorem #2. (presented in Section 2.2.1, on page 36.) Hint: Your proof should be very much like your proof in exercise [16] of Section 1.4. That is, use one of the axioms to state that any two lines intersect. Then assume that the lines intersect more than once, and show that you reach a contradiction.

[8] Justify the steps in the proof of Fano's Geometry Theorem #3. (presented in Section 2.2.2)

[9] Justify the steps in the proof of Fano's Geometry Theorem #4. (presented in Section 2.2.3)

[10] In Fano's Geometry, do parallel lines exist?

[11] In Fano's Geometry, given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L?

[12] Prove that Fano's axioms are independent. Hint: It is fairly easy to prove that axioms <F1> through <F4> are independent. Proving that axiom <F5> is independent can seem daunting. You must come up with an interpretation in which the statements of axioms <F1>, <F2>, <F3>, <F4> are true, but the statement of axiom <F5> is false. My advice is that you postpone this part of the question until you read about Young's Geometry. And you might want to do this problem at the same time that you do exercise [13].

Exercises [13] - [16] explore Young's Geometry, introduced in Section 2.2.6 on page 40.

[13] Are Young's axioms consistent? Explain.

[14] Prove that Young's axioms are independent. Hint: See exercises [12] and [13].

[15] Are there parallel lines in Young's Geometry? Explain.

[16] In Young's Geometry, given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L? Explain.

Exercises for Section 2.3 Incidence Geometry

[17] Justify the steps in the proof of Incidence Geometry Theorem #1. (presented in Section 2.3.3 on page 42.)

[18] Justify the steps in the proof of Incidence Geometry Theorem #2. (presented in Section 2.3.4 on page 43.)

[19] Justify the steps in the proof of Incidence Geometry Theorem #3. (presented in Section 2.3.5 on page 44.)

[20] Justify the steps in the proof of Incidence Geometry Theorem #5. (presented in Section 2.3.7 on page 46.)

[21] Prove that Fano's Geometry (presented in Section 2.2.1) is a model of Incidence Geometry. Is it a concrete model or an abstract model? Explain.

[22] Prove that Young's Geometry (presented in Section 2.2.6) is a model of Incidence Geometry. Hint: The proof that works for [16] should also work for this problem. Is it a concrete model or an abstract model? Explain.
[23] Prove Incidence Geometry Theorem #1, which was presented in Section 2.3.3 on page 42.

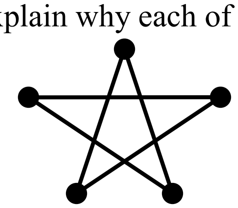
[24] Prove Incidence Geometry Theorem #2, which was presented in Section 2.3.4 on page 43.

[25] Prove Incidence Geometry Theorem #3, which was presented in Section 2.3.5 on page 44.

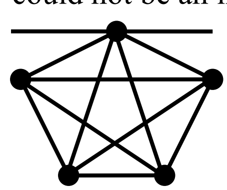
[26] Prove Incidence Geometry Theorem #4, which was presented in Section 2.3.6 on page 44.

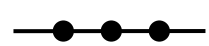
[27] Prove Incidence Geometry Theorem #5, which was presented in Section 2.3.7 on page 46.

[28] Explain why each of the pictures below could not be an incidence geometry.


picture (a)


picture (b)


picture (c)


picture (d)

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Exercises for Section 2.4 Advanced Topic: Duality

Here is a new axiom system, along with a theorem that we won't prove, but will assume as true:

Axiom System:	Five-Point Geometry
Primitive Objects:	*point*, *line*

Primitive Relations:	relation from the set of all *points* to the set of all *lines*, spoken "*The point lies on the line*".
Axioms:	<1> There are five *points*. These may be denoted P_1, P_2, P_3, P_4, P_5. <2> For any two distinct *points*, there is exactly one *line* that both *points* lie on. <3> For any line, there exist exactly two points that lie on the line.

Five-Point Geometry Theorem #1: There are exactly ten lines.

[29] Write down the axiom system for the Dual of Five-Point Geometry.

[30] How many points exist in the Dual of Five-Point Geometry? Explain.

Review Exercise for Chapters 1 and 2

[31] (A) The goal is to make up a new geometry that is consistent but not complete. I have provided the primitive terms and and relations. You provide the list of axioms.

Axiom System:	Example of a Geometry that is Consistent but not Complete
Primitive Objects:	*point*, *line*
Primitive Relations:	"*The point lies on the line*".
Axioms:	You make up the list of axioms.

(B) Explain how you know that the geometry is consistent.
(C) Explain how you know that the geometry is not complete.

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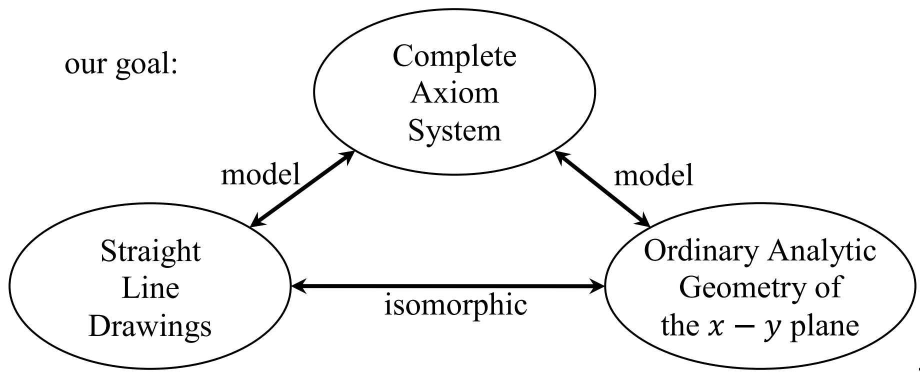
3.Neutral Geometry I: The Axioms of Incidence and Distance

We want an axiom system to prescribe the behavior of points and lines in a way that accurately represents the “straight line” drawings that we have been making throughout our lives. Remember, though, that in the language of axiom systems we turn the idea of representation around and say that we want the “straight line” interpretation to be a model of our axiom system. And because our “straight line” drawings and the usual Analytic Geometry of the $x - y$ plane behave the same way, we would expect that the usual Analytic Geometry would also be a model. But we want those to be the only models of our axiom system. Or rather, we want any other models of the axiom system to be isomorphic to the Analytic Geometry Model. That is, we would like our axiom system to be complete.



3.1. Neutral Geometry Axioms and First Six Theorems

In Chapters 3 through 8,we will study an axiom system called Neutral Geometry. It has ten axioms, and we will see that it captures much of the behavior that we are used to seeing in our straight-line drawings and in Analytic Geometry of the $x - y$ plane. However, later in the course, we will discuss the fact that the axiom system for Neutral Geometry is incomplete. An eleventh axiom will be added to make the axiom system complete, and the resulting axiom system will be called Euclidean Geometry. But that is far in the future. For now, we will study Neutral Geometry. Here is the definition.

Definition 17 The Axiom System for Neutral Geometry

Primitive Objects: *point*, *line*

Primitive Relation: *the point lies on the line*

Axioms of Incidence and Distance

<N1> There exist two distinct points. (at least two)

<N2> For every pair of distinct points, there exists exactly one line that both points lie on.

<N3> For every line, there exists a point that does not lie on the line. (at least one)

<N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$, called the Distance Function on the Set of Points.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Axiom of Separation

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following three properties:
(i) The three sets L, H_1, H_2 form a partition of the set of all points.
(ii) Each of the half-planes is convex.
(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

Axioms of Angle measurement

<N7> (Angle Measurement Axiom) There exists a function $m: \mathcal{A} \rightarrow (0,180)$, called the Angle Measurement Function.

<N8> (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $m(\angle PAB) = r$.

<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Axiom of Triangle Congruence

<N10> (*SAS* Axiom) If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

In this chapter, we will study only the first five axioms, the so-called *Axioms of Incidence and Distance*. Notice that we have seen Axioms <N1>, <N2>, and <N3> in previous axiom systems. In particular, all three of those axioms were included in the axiom system called *Incidence Geometry* that we studied in Section 2.3.2, The Axiom System for Incidence Geometry, beginning on page 41. In that section, you saw the presentation of Incidence Geometry Theorems #1 through #5, which could be proven using just those three axoms. It is convenient, then, to use those five theorems as the first five theorems of our new Neutral Geometry. That is, those first five theorems use only the first three Neutral Geometry axioms, and you have already seen a discussion of the proofs of the theorems, back in Section 2.3.2. Here are the theorems, presented now as theorems of Neutral Geometry:

Theorem 1 In Neutral Geometry, if L and M are distinct lines that intersect, then they intersect in only one point.

Theorem 2 In Neutral Geometry, there exist three non-collinear points.

Theorem 3 In Neutral Geometry, there exist three lines that are not concurrent.

Theorem 4 In Neutral Geometry, for every point P, there exists a line that does not pass through P.

Theorem 5 In Neutral Geometry, for every point P, there exist at least two lines that pass through P.

Notice the theorem numbering. This is the beginning of a sequence of theorems that ends with Theorem 160 in Chapter 14.

There is Some Subtlety in the Issue of a Line L Through Points P and Q.

Consider the following question: Given any points P and Q, is there a line that passes through P and Q, and is it unique? This seems like an easy question. Axiom <N2> clearly states "Given any two distinct points, there is exactly one line that both points lie on." But notice that the axiom includes the qualifier *two distinct points*, and my question did not include that qualifier.

Certainly, if it is known that points P and Q are distinct, then Axiom <N2> guarantees that there exists exactly one line that both points lie on. But what if points P and Q are not distinct? That is, what if point Q is actually the same point as point P? In this case, is there a line that passes through P, and is it unique? Well, Theorem 5 (on page 62) says that there are at least two lines that pass through point P. So there definitely *is* a line through point P, and it definitely *is not* unique.

So in any situation where it is known that points P and Q are distinct, it makes sense to talk of *the unique line that passes through* points P and Q. Here is a symbol that we can use for that:

Definition 18 the unique line passing through two distinct points
words: line P, Q
symbol: $\overleftrightarrow{PQ}$
usage: P and Q are distinct points
meaning: the unique line that passes through both P and Q. (The existence and uniqueness of such a line is guaranteed by Axiom <N2>.)

In any situation where it is *not* known if points P and Q are distinct, it is okay to talk of *a line that passes through* points P and Q, but one must not assume that such a line is unique, and one must not assume that such a line is guaranteed by Axiom <N2>. If it turns out that points P and Q are distinct, then the existence of the line is guaranteed by Axiom <N2> and the line is unique; If it turns out that points P and Q are not distinct, then the existence of the line is guaranteed by Theorem 5, not by Axiom <N2>, and the line is definitely not unique.

The previous paragraph is summarized in the following theorem.

Theorem 6 In Neutral Geometry, given any points P and Q that are not known to be distinct, there exists at least one line that passes through P and Q.

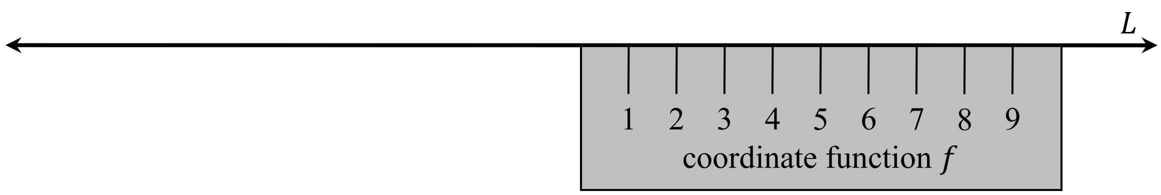
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 3.11 on page 94.

3.2. The Distance Function and Coordinate Functions in Drawings

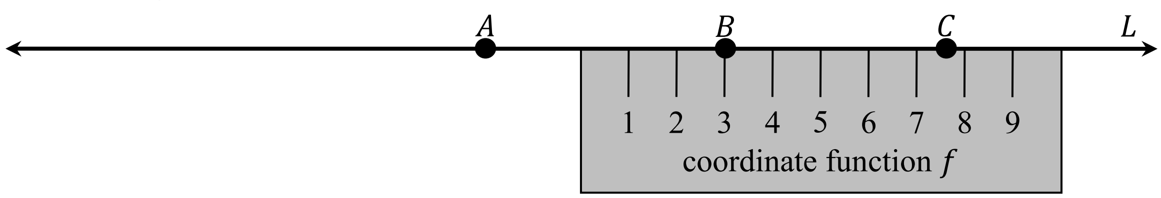
In the finite geometries that we have studied so far, there has been no mention of distance. In this book, we are studying Abstract Neutral and Euclidean Geometry. (In Chapters 3 through 8, we study Abstract Neutral Geometry. Our study of Euclidean Geometry begins in Chapter 9.) I have said that our axiomatic geometry is to be an abstract version of the "straight-line" drawings that we have been making all of our lives. In those drawings, we are able to measure distance. So, our axiomatic geometry must include some axioms or theorems that specify how that is done.

Before we study the axioms about distance in Neutral Geometry (a study that begins in Section 3.4, The Distance Function and Coordinate Functions in Neutral Geometry), it will be helpful to consider what we do when we measure distance in drawings. In the current section, we will study that process of measuring distance in drawings and to try to describe the process with some precision.
Consider the drawn line L shown below.

In the next drawing, below, I have put a ruler alongside the line. The ruler can be used to assign a real number to each point on the line. That is, when placed alongside the line, the ruler can be thought of as a function with domain the set of points on line L and codomain the set of real numbers. Such a function is called a coordinate function. We could give this coordinate function the name f. In symbols, $f: L \to \mathbb{R}$.



To see this coordinate function f in action, consider the three drawn points A, B, C shown on line L in the drawing below.



If we use the three drawn points A, B, C as input to the function f, the resulting outputs are three real numbers, called the *coordinates* of the three points.

- The coordinate of drawn point A is the real number $f(A) = -2$.
- The coordinate of drawn point B is the real number $f(B) = 3$.
- The coordinate of drawn point C is the real number $f(C) = 7.6$.

Notice that we have had to do some abstract thinking when making these measurements. For one, the drawn point A is off the end of the ruler. We had to imagine the ruler extending beyond what is shown. Also, the point C lies between two marks on the ruler. We had to imagine subdivisions on the ruler. So there is a certain amount of abstraction in our use of ordinary drawing tools.

What about measuring distance? Using the technique that we have used since grade school, we would say that the distance between A and B is 5, while the distance between C and B is 4.6. A precise description of how we got these numbers is as follows

- The distance between A and B is $|f(A) - f(B)| = |(-2) - 3| = |-5| = 5$.
- The distance between C and B is $|f(C) - f(B)| = |(7.6) - 3| = |4.6| = 4.6$.

So it seems that in drawings we can simply define the distance between two points as follows.

To find the distance between drawn points A and B:
- Put a ruler alongside the line L that passes through points A and B.
- The placement of the ruler alongside the line gives us a coordinate function $f: L \to \mathbb{R}$. The coordinates of points A and B are the real numbers $f(A)$ and $f(B)$.
- The distance between points A and B is defined to be the real number $|f(A) - f(B)|$.

The process just described for measuring distance between drawn points can be thought of as a function. The input to the function is a pair of drawn points. The output of the function is the real number called *the distance between the points*. We could call this function *the distance function for drawn points*.

In this section, we studied the process of measuring distance in drawings, describing the process using function terminology. The overall process was given the name *distance function*. But the process involved a couple of steps. Two of those steps were to place a ruler alongside the line and read numbers from the ruler. That part of process was given the name *coordinate function*. In the next section, we will study the process of measuring distance in Analytic Euclidean Geometry. We will see the familiar *distance function* for Analytic Geometry. In addition, we will see that it is possible to define *coordinate functions* for Analytic Geometry.

3.3. The Distance Function and Coordinate Functions in Analytic Geometry

In Analytic Euclidean Geometry, points are ordered pairs of real numbers (x, y). The set of all points is the set of all ordered pairs or real numbers, denoted by the symbol $\mathbb{R} \times \mathbb{R}$ or by the more abbreviated symbol $\mathbb{R}^2$.

In Analytic Euclidean Geometry, lines are sets of points that satisfy line equations. That is, equations of the form $ax + by = c$ where a, b, c are real number constants. For example, a line L could be described as follows.

$$L = \{(x, y) \text{ } such \text{ } that \text{ } 2x + 3y = 5\}$$

In Analytic Euclidean Geometry, the distance function is the function $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by the following equation

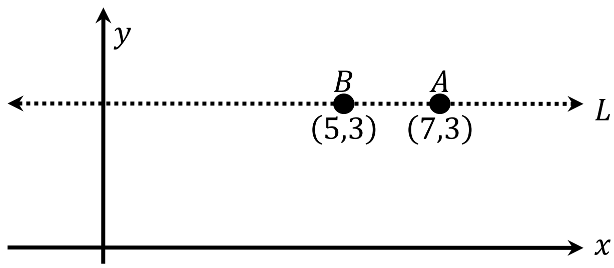
$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The concepts of points, lines, and the distance function in Analytic Geometry are not new to you. But you have probably never seen something called a *coordinate function* in Analytic Geometry. That is mainly because your prior school work in Analytic Geometry has not needed coordinate functions. We won't need them in this book, either, but we will study them so that you can see that they exist and behave in a way analogous to the coordinate functions that we introduced for drawings. We will consider two lines called L and M, and will see the introduction of coordinate functions for those lines.

Let L be the line described as follows.

$$L = \{(x, y) \text{ such that } y = 3\}$$

Line L is horizontal, parallel to the x-axis but 3 units above the x-axis. Let A and B be the two points $A = (7,3)$ and $B = (5,3)$. Both points are on line L.



Observe that the distance between A and B is

$$d(A, B) = d\big((7,3), (5,3)\big) = \sqrt{(5-7)^2 + (3-3)^2} = \sqrt{(-2)^2 + (0)^2} = 2$$

Define the function $f: L \to \mathbb{R}$ by the equation

$$f(x, y) = x$$

Using points A and B as inputs to the function f, we obtain the following outputs.

- When point A is the input, the output is the real number $f(A) = f(7,3) = 7$.
- When point B is the input, the output is the real number $f(B) = f(5,3) = 5$.

Also observe that

$$|f(A) - f(B)| = |7 - 5| = |2| = 2$$

so the equation

$$|f(A) - f(B)| = d(A, B)$$

is satisfied. The equation tells us that the real number $|f(A) - f(B)|$ is equal to the distance between points A and B.The fact that this equation is satified is what qualifies f to be called a *coordinate function* for line L. (Recall that in the previous section, when discussing rulers and distance in drawings, we observed that in drawings the real number $|f(A) - f(B)|$ is equal to the distance between points A and B. So the relationship between coordinate functions and distance in Analytic Geometry is the same as the relationship between coordinate functions and distance in drawings.)

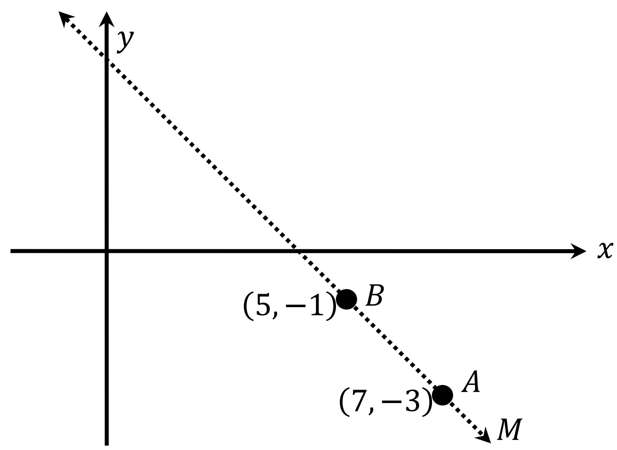
We can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function f, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $f(A) = f(7,3) = 7$.
- The coordinate of point B is the real number $f(B) = f(5,3) = 5$.

Now consider the slanting line M described as follows.

$$M = \{(x, y) \text{ } such \text{ } that \text{ } x + y = 4\}$$

That is, $y = -x + 4$. Let A and B be the two points $A = (7, -3)$ and $B = (5, -1)$ on line M.



Observe that the distance between A and B is

$$d(A, B) = d\big((7, -3), (5, -1)\big) = \sqrt{(5-7)^2 + \big(-1-(-3)\big)^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

Suppose that we define a function $f: M \to \mathbb{R}$ by the equation

$$f(x, y) = x$$

This function f looks just like the function f that was a coordinate function for line L. Let's investigate to see if the function f could be a coordinate function for line M. Using points A and B as inputs to the function f, we obtain the following outputs.

- When point A is the input, the output is the real real number $f(A) = f(7, -3) = 7$.
- When point B is the input, the output is the real real number $f(B) = f(5, -1) = 5$.

Observe that

$$|f(A) - f(B)| = |7 - 5| = |2| = 2$$

So the equation

$$|f(A) - f(B)| = d(A, B)$$

is *not* satisfied! For this reason, we say that f *is not* qualified to be called a *coordinate function* for line M.

It is useful to examine what was wrong with our function f above. Notice the following values:

$$d(A,B) = 2\sqrt{2}$$
$$|f(A) - f(B)| = 2$$

The two values are off by a factor of $\sqrt{2}$. That gives us an idea for how we might change the definition of function f in order to make it qualify to be called a coordinate function. We can define a new function $f: N \to \mathbb{R}$ by the equation
$$f(x,y) = x\sqrt{2}$$
Using points A and B as inputs to the new function f, we obtain the following outputs.

- When point A is the input, the output is the real real number $f(A) = f(7,-3) = 7\sqrt{2}$.
- When point B is the input, the output is the real real number $f(B) = f(5,-1) = 5\sqrt{2}$.

Observe that
$$|f(A) - f(B)| = \left|7\sqrt{2} - 5\sqrt{2}\right| = \left|2\sqrt{2}\right| = 2\sqrt{2}$$

So the equation
$$|f(A) - f(B)| = d(A,B)$$

is satisfied. In other words, this new function f *is* qualified to be called a *coordinate function* for line M.

As we did for earlier, we can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function f, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $f(A) = f(7,3) = 7\sqrt{2}$.
- The coordinate of point B is the real number $f(B) = f(5,3) = 5\sqrt{2}$.

In a homework exercise, you will be asked to determine which functions could be coordinate functions for a given line. And you will be asked to come up with a coordinate function of your own.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 3.11 on page 94.

3.4. The Distance Function and Coordinate Functions in Neutral Geometry

Now we will return to our study of Abstract Neutral Geometry. We will no longer be thinking of points as dots in a drawing or as ordered pairs of numbers, and we will no longer be thinking of lines as something that we draw with a ruler or as sets of ordered pairs of numbers. We will go back to having points and lines being primitive, undefined objects whose properties are specified by the axioms. And the distance function will not be the concrete function given by the equation involving the square root. We will not know a formula for the distance function. All we will know about it is that it has certain properties that we will prove in theorems

We will see that distance and coordinate functions in Neutral Geometry mimic the essential features of distance and coordinate functions in drawings and in Analytic Geometry.

The simplest way to ensure that it is possible to measure distance in an axiomatic geometry is to just include an axiom stating that a "distance function" exists. That is, indeed, the approach that is taken in our axiom system for Neutral Geometry. However, there are details to be worked out.

To begin with, we need to be clear about what it is that we are measuring the distance between. We want to measure the distance between two *points*. In order to describe the process, it is helpful to have a symbol for the set of all points.

Definition 19 The set of all abstract points is denoted by the symbol $\mathcal{P}$ and is called *the plane*.

With the above definition of the symbol $\mathcal{P}$ and with our knowledge of standard function notation from previous courses, we are now ready to understand the wording of Axiom <N4>.

> <N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

The function notation $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ tells us that the symbol d stands for a function with domain the set $\mathcal{P} \times \mathcal{P}$ of ordered pairs of points. That is, the input to the function will be a pair of the form (P, Q), where $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ are points. The function notation also tells us that the codomain is the set of real numbers. That is, when a pair of points (P, Q) is used as input to the function, the resulting output is a real number, denoted by the symbol $d(P, Q)$. The real number $d(P, Q)$ is called the distance between points P and Q.

Because the distance between two points is mentioned so often, most books adopt an abbreviated symbol for it.

Definition 20 abbreviated symbol for the distance between two points
abbreviated symbol: PQ
meaning: the distance between points P and Q, that is, $d(P, Q)$

We will not use this notation in the current chapter, but we will use it in coming chapters.

What are the properties of the Distance Function on the Set of Points, the function d? In Mathematics, the term *distance function* is usually used for a particular kind of function, one that is known (or assumed) to possess certain particular properties. In this book, the Distance Axiom <N4> tells us nothing about the properties of the Distance Function on the Set of Points. The axiom merely tells us that the function exists. Later in the book, theorems will be presented that describe the properties of the function. (The first of those theorems will show up in Section 3.6 Two Basic Properties of the Distance Function in Neutral Geometry, which starts on page 74.)

Now on to Coordinate Functions in Neutral Geometry.

As mentioned above, axiom <N4> merely states that a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ exists and is called the Distance Function on the Set of Points. The axiom does not specify how the function d

behaves. We need another axiom to give more specifics about the behavior and use of the function d. We would like the axiom to specify that the abstract distance can be measured in a manner a lot like the way that we measure distance in drawings. In drawings, we measure the distance between two points P and Q by putting a ruler alongside the two points. The ruler is used to assign a number to each point. The absolute value of the difference of the two numbers is the distance between the two points. (That process was described in Section 3.2, The Distance Function and Coordinate Functions in Drawings) In order to capture this behavior in an axiom, we will use the concept of a *coordinate function*.

Definition 21 Coordinate Function

Words: f is a coordinate function on line L.

Meaning: f is a function with domain L and codomain $\mathbb{R}$ (that is, $f: L \to \mathbb{R}$) that has the following properties:

(1) f is a one-to-one correspondence. That is, f is both one-to-one and onto.

(2) f "agrees with" the distance function d in the following way:

For all points P,Q on line L, the equation $|f(P) - f(Q)| = d(P, Q)$ is true.

Additional Terminology: In standard function notation, the symbol $f(P)$ denotes the output of the coordinate function f when the point P is used as input. Note that $f(P)$ is a real number. The number $f(P)$ is called *the coordinate of point P on line L*.

Additional Notation: Because a coordinate function is tied to a particular line, it might be a good idea to have a notation for the coordinate function that indicates which line the coordinate function is tied to. We could write f_L for a coordinate function on line L. With that notation, the symbol $f_L(P)$ would denote the coordinate of point P on line L. But although it might be clearer, we do not use the symbol f_L. We just use the symbol f.

With the above definition of *Coordinate Function*, we are now ready to understand the wording of Axiom <N5>.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Coordinate functions will play a role in our abstract geometry that is analogous to the roles played by rulers in drawings (That role was described in Section 3.2, The Distance Function and Coordinate Functions in Drawings) and by coordinate functions in Analytic Geometry (That role was described in Section 3.3, The Distance Function and Coordinate Functions in Analytic Geometry). The analogy will be explored more in coming sections. But before going on to that, we should point out that simply knowing that each line has a coordinate function tells us something very important about the set of points on a line.

Theorem 7 about how many points are on lines in Neutral Geometry

In Neutral Geometry, given any line L, the set of points that lie on L is an infinite set. More precisely, the set of points that lie on L can be put in one-to-one correspondence with the set of real numbers $\mathbb{R}$. (In the terminology of sets, we would say that the set of points on line L has the same cardinality as the set of real numbers $\mathbb{R}$.)

This theorem is worth discussing a bit. First, contrast what the theorem says with what we knew about lines in some of our previous geometries in Chapter 2.

- Four Point Geometry
    - Four Point Axiom <3> For any line, there exist exactly two points that lie on the line.
- Four Line Geometry
    - Four-Line Geometry Theorem #2: For every line, there exist exactly three points that the line passes through.
- Fano's Geometry
    - Fano's Axiom <F2> For every line, there exist exactly three points that lie on the line.
- Young's Geometry
    - Young's Axiom <Y2> For every line, there exist exactly three points that lie on the line.
- Incidence Geometry
    - Incidence Axiom <I4> For every line, there exist two points that lie on the line (at least two points)

We see that most of the geometries that we studied in Chapter 2 had lines with only a finite number of points lying on them. Incidence Geometry is the only geometry that we studied in Chapter 2 that might have lines with an infinite set of points lying on them. But that was not a requirement. That is, we saw examples of Incidence Geometries that had lines with only a finite number of points.

Also notice that Theorem 7 (about how many points are on lines in Neutral Geometry), found on page 70, is very different from Neutral Axiom <N2>. That axiom says that given any two distinct points, there is exactly one line that both points lie on. The axiom does not say that given any line, there are exactly two points that lie on it. We see that the second statement is not even true. That is, Theorem 7 tells us that in Neutral Geometry, given any line, the set of points that lie on it is an infinite set.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 3.11 on page 94.

3.5. Diagram of Relationship Between Coordinate Functions & Distance Functions

In Section 3.4 (The Distance Function and Coordinate Functions in Neutral Geometry), which started on page 68, we discussed the function d called the Distance Function on the Set of Points. This is a function that takes as input a pair of *points* and produces as output a real number called the distance between the points. It is illuminating to think about a different distance function on a different set, the set of real numbers, $\mathbb{R}$.

We are all familiar with the idea that to find the distance between two numbers on the number line, one takes the absolute value of their difference. That is, the distance between x and y is $|x-y|$. We want to describe this using the terminology and notation of functions. That is, we would like to say that the calculation $|x-y|$ describes the working of some function.

We will call the function a "distance function", but we must be careful because we already discussed something called a distance function when we were discussing the Distance Axiom <N4>. The distance function in our current discussion is not the same one as the one mentioned in axiom <N4>. The distance function mentioned in axiom <N4> measures the distance between two points and is called the Distance Function on the Set of Points. Our current distance function measures the distance between real numbers, so we will call it the Distance Function on the Set of Real Numbers and will denote it by the symbol $d_{\mathbb{R}}$.

The domain of the Distance Function on the Set of Points, the function d, is the set $\mathcal{P} \times \mathcal{P}$ of ordered pairs of points. In the current discussion, we are measuring the distance between two real numbers, so the domain of the Distance Function on the Set of Real Numbers, the function $d_{\mathbb{R}}$, will be the set $\mathbb{R} \times \mathbb{R}$ of ordered pairs of real numbers.

Given any two real numbers, the distance between them is a real number. In the terminology of functions, we say that the codomain of the Distance Function on the Set of Real Numbers, the function $d_{\mathbb{R}}$, is the set of Real Numbers, $\mathbb{R}$.

Using the terminology and symbols of the preceeding discussion, we are ready to state a definition.

Definition 22 Distance Function on the set of Real Numbers
Words: The Distance Function on the Set of Real Numbers
Meaning: The function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$.

For examples of the use of the Distance Function on the Set of Real Numbers, consider the following calculations.

$$d_{\mathbb{R}}(5,7) = |5 - 7| = |-2| = 2$$
$$d_{\mathbb{R}}(7,5) = |7 - 5| = |2| = 2$$
$$d_{\mathbb{R}}(5,5) = |5 - 5| = |0| = 0$$
$$d_{\mathbb{R}}(-5,-7) = |(-5) - (-7)| = |2| = 2$$

Having introduced the Distance Function on the Set of Real Numbers, the function $d_{\mathbb{R}}$, we now will use the function $d_{\mathbb{R}}$ to shed new light on the Distance Function on the Set of Points, the function d.

Observe that given points P and Q, there are two different processes that can be used to produce a real number.

Process #1:

Feed the pair of points (P, Q) into the Distance Function on the Set of Points, the function d, to get a real number, denoted $d(P, Q)$ called the distance between P and Q. This process could be illustrated with an arrow diagram:

$$\begin{array}{ccc} (P,Q) & \longmapsto & d(P,Q) \\ & d & \\ \mathcal{P} \times \mathcal{P} & \longrightarrow & \mathbb{R} \end{array}$$

The bottom half of the diagram, we have seen before. It is the arrow diagram that tells us that the symbol d represents a function with domain $\mathcal{P} \times \mathcal{P}$ and codomain $\mathbb{R}$. The top part of the diagram has been added. It shows what happens to an actual pair of points.

Process #2: (This is a two-step process.)

First Step: Let L be a line passing through points P and Q and let f be a coordinate function for line L. Feed point P into f to get a real number $f(P)$, and feed point Q into f to get a real number $f(Q)$. This gives us a pair of real numbers, $\big(f(P), f(Q)\big)$.

Second Step: Feed the pair of real numbers $\big(f(P), f(Q)\big)$ into the Distance Function on the Set of Real Numbers, the function $d_{\mathbb{R}}$, to get a real number, denoted $d_{\mathbb{R}}\big(f(P), f(Q)\big)$. We know exactly how the Distance Function on the Set of Real Numbers works. The real number $d_{\mathbb{R}}\big(f(P), f(Q)\big)$ is just $|f(P) - f(Q)|$.

The two-step process can be illustrated with a two-step arrow diagram:

$$\begin{array}{ccccc} (P,Q) & \longmapsto & \big(f(P),f(Q)\big) & \longmapsto & |f(P)-f(Q)| \\ \mathcal{P}\times\mathcal{P} & \xrightarrow{\ f\times f\ } & \mathbb{R}\times\mathbb{R} & \xrightarrow{\ d_{\mathbb{R}}\ } & \mathbb{R} \end{array}$$

So we have two different processes that can be used to turn a pair of points into a single real number. An obvious question is this: do the two processes give the same result? That is, for any points P and Q and a coordinate function f on a line L passing through P and Q, does $d(P,Q)$ equal $|f(P) - f(Q)|$? Well, the fact that f is a coordinate function guarantees that the two results will always match.

$$d(P,Q) = |f(P) - f(Q)|$$
$$result\ of\ process\ \#1 = result\ of\ process\ \#2$$

The fact that these two processes always yield the same result can be illustrated by combining the two arrow diagrams into a single, larger diagram. In order to improve readability, we will bend the diagram for process #2. The resulting diagram is

$$\begin{tikzcd}
\mathcal{P}\times\mathcal{P} \arrow[r, "f\times f"] \arrow[dr, "d"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "d_{\mathbb{R}}"] \arrow[dl, phantom, "\circledequal"] \\
& \mathbb{R}
\end{tikzcd}$$

In the diagram, we see that there are two different routes to get from a pair of points (that is, an element of $\mathcal{P} \times \mathcal{P}$) to the set of real numbers, $\mathbb{R}$. The slanting arrow is Process #1. The two-step path that goes straight across and then straight down is Process #2. The circled equal sign in the middle of the diagram indicates that these two paths always yield the same result. In diagram jargon, we say that the diagram *commutes*.

We can superimpose on the diagram some additional symbols that show what happens to an actual pair of points.

$$\begin{tikzcd}
(P,Q) \arrow[rrr, mapsto] \arrow[dddrr, mapsto] & & & \big(f(P),f(Q)\big) \arrow[dd, mapsto] \\
& \mathcal{P}\times\mathcal{P} \arrow[r, "f\times f"] \arrow[dr, "d"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "d_{\mathbb{R}}"] \arrow[dl, phantom, "\circledequal"] & \\
& & \mathbb{R} & |f(P)-f(Q)| \\
& & d(P,Q) &
\end{tikzcd}$$

The two diagrams above may seem rather strange to you, but these sorts of diagrams are very common in higher-level math. Remember that the two diagrams are merely illustrations of what it means when we say that a function f is a coordinate function. They illustrate the relationship between a coordinate function f and the distance function d.

3.6. Two Basic Properties of the Distance Function in Neutral Geometry

As mentioned in Section 3.4 (The Distance Function and Coordinate Functions in Neutral Geometry), which started on page 68,the Distance Axiom <N4> tells us that the Distance Function on the Set of Points, the function d, exists, but the axiom does not tell us anything about the properties of that function. All of the facts that we want to state about the properties of the Distance Function on the Set of Points will have to be proven in theorems. Most of those theorems will be stated and proven in the current chapter, using proofs that rely on the important relationship between the Distance Function and Coordinate Functions. That relationship was introduced in the definition of Coordinate Function (Definition 21 on page 70) and was illustrated in diagrams in the previous section.

For all points P,Q on line L, the equation $|f(P) - f(Q)| = d(P,Q)$ is true.

In this section, we will prove that d has two important properties.

The first property of d that we will prove is the property of being *positive definite*.

Theorem 8 The Distance Function on the Set of Points, the function d, is Positive Definite. For all points P and Q, $d(P,Q) \geq 0$, and $d(P,Q) = 0$ if and only if $P = Q$. That is, if and only if P and Q are actually the same point.

Proof

(1) Let P and Q be any two points, not necessarily distinct.
(2) Let L be a line passing through P and Q. (The existence of such a line is guaranteed by Theorem 6 (In Neutral Geometry, given any points P and Q that are not known to be distinct, there exists at least one line that passes through P and Q.) found on page 63.)
(3) Let f be a coordinate function for line L. (A coordinate function exists by axiom <N5>.)
(4) By the definition of Coordinate Function (Definition 21 on page 70), we know that $d(P,Q) = |f(P) - f(Q)|$. This tells us that $d(P,Q) \geq 0$.
(5) Suppose that P and Q are actually the same point. Then their coordinates $f(P)$ and $f(Q)$ will be the same real number, so $d(P,Q) = |f(P) - f(Q)| = 0$.
(6) Now suppose that P and Q are not the same point. Because coordinate functions are one-to-one, the coordinates $f(P)$ and $f(Q)$ will not be the same real number. So the difference $|f(P) - f(Q)|$ will not be zero. Therefore, $d(P,Q) = |f(P) - f(Q)| \neq 0$.
(7) The previous two steps tell us that $d(P,Q) = 0$ if and only if $P = Q$.

End of proof

The second property of d that we will prove is the property of being *symmetric*.

Theorem 9 The Distance Function on the Set of Points, the function d, is Symmetric. For all points P and Q, $d(P,Q) = d(Q,P)$.

Proof

Let P and Q be any two points. Let L be a line passing through P and Q, and let f be a coordinate function for line L.

$$\begin{aligned} d(P,Q) &= |f(P) - f(Q)| \ (\boldsymbol{justify}) \\ &= |f(Q) - f(P)| \\ &= d(Q,P) \quad (\boldsymbol{justify}) \end{aligned}$$

End of proof

The two properties of the Neutral Geometry distance function that we have studied in this section--the fact that the distance function is positive definite and symmetric--are properties that will be important to us in future proofs, but they are both properties that you have probably never thought about or used when measuring distance in drawings or in Analytic Geometry.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 3.11 on page 94.

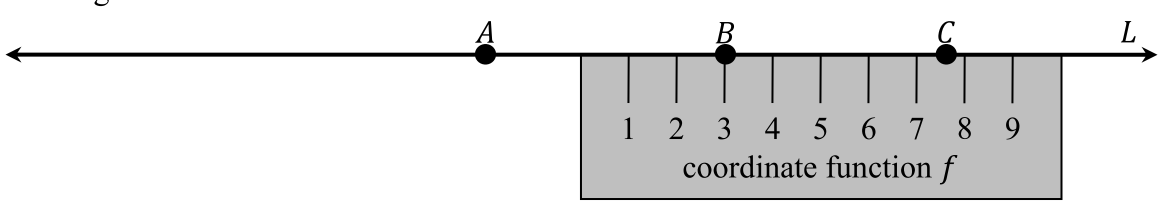
3.7. Ruler Placement in Drawings

As mentioned in Section 3.4, coordinate functions in Neutral Geometry are defined in a way that seems to mimic the role played by rulers and coordinate functions in drawings. But in that section, we did not explore the behavior of coordinate functions very much. In the next few sections, we will study the concept of *ruler placement* in drawings, in Analytic Geometry, and in Neutral Geometry. Although you have probably not used the name *ruler placement* before, the term does describe things that you have certainly done when using a ruler to measure things.

To be more precise, in this section we will study three behaviors of rulers in drawings: Ruler Sliding, Ruler Flipping, and Ruler Placement. In Section 3.8 (Ruler Placement in Analytic Geometry, starting on page 81) and in Section 3.9 (Ruler Placement in Neutral Geometry, starting on page 87), we will see that there are analogs of Ruler Sliding, Ruler Flipping, and Ruler Placement in Analytic Geometry and in Neutral Geometry, as well.

First Behavior of Rulers in Drawings: Ruler Sliding

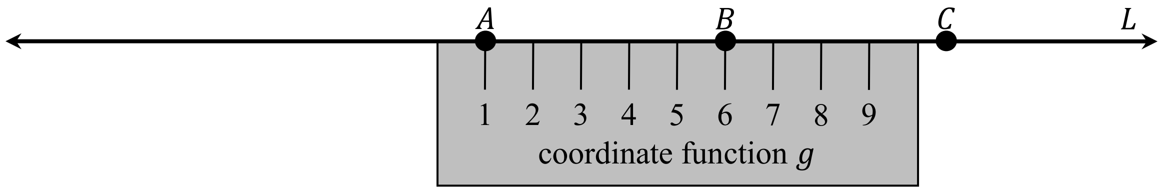
The first behavior of rulers in drawings that we will consider is that of sliding the ruler along the line. In Section 3.2 (The Distance Function and Coordinate Functions in Drawings, starting on page 63), we put a ruler alongside a drawn line L. The placement of the ruler alongside the line gave us a coordinate function that we called f, In symbols, we wrote $f: L \to \mathbb{R}$. To see this coordinate function f in action, we considered the three drawn points A, B, C shown on line L in the drawing below.



If we use the three drawn points A, B, C as input to the function f, the resulting outputs are three real numbers, called the *coordinates* of the three points.

- The coordinate of drawn point A is the real number $f(A) = -2$.
- The coordinate of drawn point B is the real number $f(B) = 3$.
- The coordinate of drawn point C is the real number $f(C) = 7.6$.

Consider what happens if we use the same ruler that we used above, but slide it three units to the left. ("Right" is the direction of increasing numbers; "left" is the direction of decreasing numbers.) The result is a new coordinate function that we can call g. In symbols, $g: L \to \mathbb{R}$.



If we use the three points A, B, C as input to the coordinate function g, the resulting outputs are three real numbers listed below.

- The coordinate of drawn point A is the real number $g(A) = 1$.
- The coordinate of point B is the real number $g(B) = 6$.
- The coordinate of point C is the real number $g(C) = 10.6$.

Using the coordinate function g, we find that the distance between A and B is 5 and the distance between C and B is 4.6. So the distances between the points are unchanged. Good.

Notice that for any point P on line L, the coordinate obtained using coordinate function g is always going to be 3 greater than the coordinate obtained using coordinate function f. That is,

$$g(P) = f(P) + 3$$

More generally, if we slide ruler along line L by some amount, we will obtain a new coordinate function that we could call coordinate function g, and the coordinates produced by the two coordinate functions f and g would be related by the equation

$$g(P) = f(P) + c$$

where c is a real number constant that will depend on how far the ruler was slid. (And which way it was slid! If we slide the ruler to the right, then the constant c will be a negative number.)

We could also think of this the other way around in the following sense: Suppose we are given a real number -7. Is there a way to place the ruler so that it will provide a coordinate function g with the property that for any point P on line L,

$$g(P) = f(P) + (-7) = f(P) - 7?$$

Of course there is. To place the ruler for coordinate function g, one should slide the ruler 7 units to the right of the spot that it was in for coordinate function f.

More generally, given any real number c, the equation

$$g(P) = f(P) + c$$

describes a new coordinate function that can be achieved by sliding the ruler that produced coordinate function f to some new spot alongside line L.

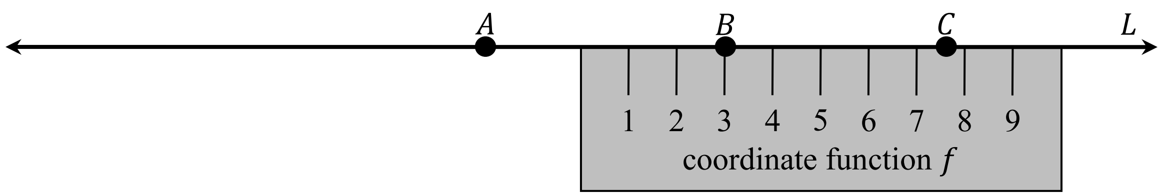
Let's take a moment to briefly look ahead at something found in Section 3.9 (Ruler Placement in Neutral Geometry) . The Theorem 10 claim (A) on page 87 says.

Suppose that $f: L \to \mathbb{R}$ is a coordinate function for a line L.
(A) (Ruler Sliding) If c is a real number constant and g is the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) + c$, then g is also a coordinate function for line L.

We see that Theorem 10 claim (A) is simply telling us that our abstract coordinate functions will exhibit behavior analogous to the sliding behavior of drawn rulers discussed above.

Second Behavior of Rulers in Drawings: Ruler Flipping

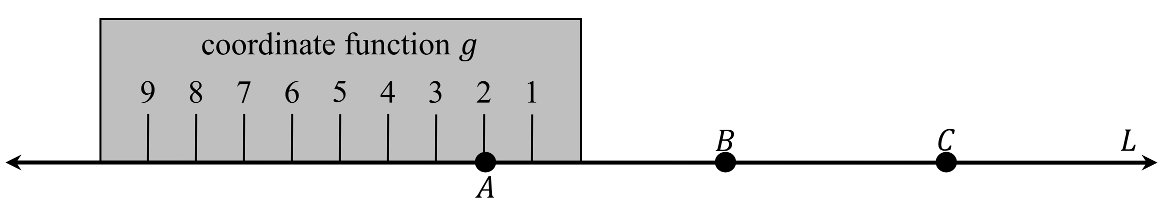
The second behavior of rulers in drawings that we will consider is that of flipping the ruler.We start by reviewing the placement of the ruler that gave us coordinate function f.



If we use the three drawn points A, B, C as input to the function f, the resulting outputs are three real numbers, called the *coordinates* of the three points.

- The coordinate of drawn point A is the real number $f(A) = -2$.
- The coordinate of drawn point B is the real number $f(B) = 3$.
- The coordinate of drawn point C is the real number $f(C) = 7.6$.

Now flip the ruler around, keeping the zero in the same spot on the line, so that the numbers go to the left. (My drawing program won't let me flip the characters upside down.) This placement gives us a new coordinate function, one that we can call g.



If we use the three drawn points A, B, C as input to the function g, the resulting outputs are three real numbers, called the *coordinates* of the three points.

- The coordinate of drawn point A is the real number $g(A) = 2$.
- The coordinate of drawn point B is the real number $g(B) = -3$.
- The coordinate of drawn point C is the real number $g(C) = -7.6$.

Notice that for any point P on line L, the coordinate obtained using coordinate function g is always going to the negative of the coordinate obtained using coordinate function f. That is,

$$g(P) = -f(P)$$

Let's again take a moment to briefly look ahead at something found in Section 3.9 (Ruler Placement in Neutral Geometry) . The Theorem 10 claim (B) on page 87 says.

Suppose that $f\colon L \to \mathbb{R}$ is a coordinate function for a line L.
(B) (Ruler Flipping) If c is a real number constant and g is the function $g\colon L \to \mathbb{R}$ defined by $g(P) = -f(P)$, then g is also a coordinate function for line L.

We see that Theorem 10 claim (B) is simply telling us that our abstract coordinate functions will exhibit behavior analogous to the flipping behavior of drawn rulers discussed above.

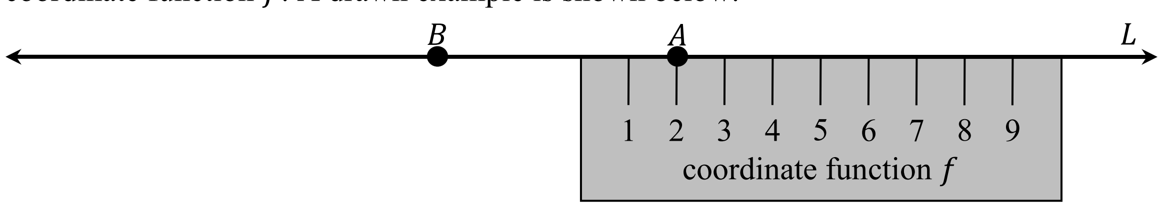
Combining Sliding and Flipping: Ruler Placement in Drawings

The third behavior of rulers in drawings that we will consider is that of ruler placement. That name may sound vague and unhelpful, but the name refers to the following simple idea. Given two drawn points in a drawing, it is often useful to put the end of a ruler on one of the points, and have the numbers on the ruler go in the direction of the other point. That is, given points A and B on a line, we often want the ruler placed so that the resulting coordinate function has two properties:

- the coordinate of point A is zero
- the coordinate of point B is positive

With an actual ruler and a drawing, we can simply put the ruler on the drawing in the right way. But it will be helpful to describe the process of getting the ruler into the correct position in a more abstract way, so that we may see that the process generalizes to the abstract coordinate functions of Neutral Geometry. We can get the ruler into the right position by a combination of sliding and flipping.

Suppose that we are given points A and B lying on some line L. Put a ruler alongside line L. Placed alongside the line like this, the ruler gives us a coordinate function for line L. We can call the coordinate function f. A drawn example is shown below.



Notice that in our drawn example,

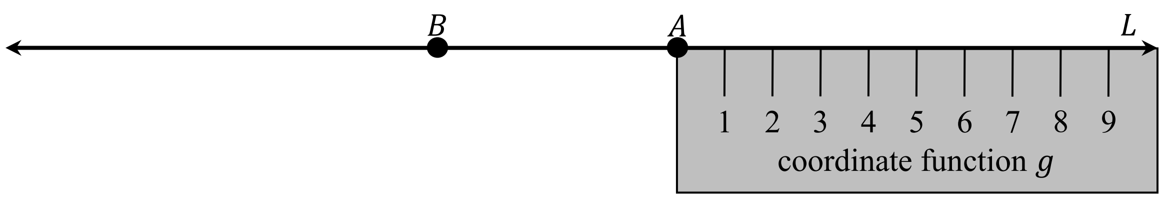
$$f(A) = 2$$
$$f(B) = -3$$

So coordinate function f *does not* have either of the properties that we desire.

Now we will move the ruler to obtain a coordinate function that *does* have our desired properties. It will be done in two steps.

Step 1: Ruler Sliding

Let c be the real number defined by $c = f(A)$. That is, c is the real number coordinate of point A using coordinate function f on line L. In our drawn example, $c = f(A) = 2$. Move the ruler c units to the right along line L. ("Right" is the direction of increasing numbers on the ruler. Also note that if the number c is negative, then "moving the ruler c units to the right" will actually mean that the ruler gets moved to the left. For example, if $c = -7$, then moving the ruler $c = -7$ units to the right will mean that the ruler actually gets moved 7 units to the left.) This new placement of the ruler gives us a new coordinate function for line L. We can call the new coordinate function g. The result of doing this in our drawn example is shown below, where we have moved the ruler 2 units to the right.



Notice that in our drawn example,

$$g(A) = 0$$
$$g(B) = -5$$

So coordinate function g in our drawn example has the first property that we desire, but not the second property.

This behavior of the coordinate function g can be described abstractly. Because the ruler was moved c units to the right, the coordinates produced by functions f and g will be related by the equation

$$g(P) = f(P) - c$$

In this equation, the letter P represents an arbitrary point on line L. Observe that we know that the value of the constant c is just $c = f(A)$. So the coordinates produced by functions f and g will be related by the equation

$$g(P) = f(P) - f(A)$$

Again, the letter P represents an arbitrary point on line L. Notice what happens when we let $P = A$.

$$g(A) = f(A) - f(A) = 0$$

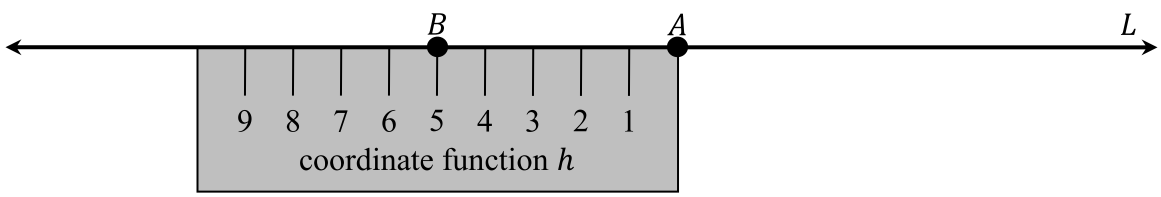
That is, the new coordinate function g has the special property that it assigns a coordinate of zero to point A. Observe that in our above drawing, $g(A) = 0$.

Step 2: Ruler Flipping

Remember that we are interested in describing abstractly how to place the ruler so that it has two properties

- the coordinate of point A is zero
- the coordinate of point B is positive.

In our drawn example, the coordinate function g has the first property, but not the second. We can fix this by simply flipping the ruler over so that the zero end remains at point A but the positive numbers go in the direction of point B. The resulting placement of the ruler gives us a new coordinate function, one that we can call h. A picture is shown below.



Notice that in our drawn example,

$$h(A) = 0$$
$$h(B) = 5$$

So coordinate function h in our drawn example has both properties that we desire.

This behavior of the coordinate function h can be described abstractly. Because the ruler was flipped so that its zero remained at the same point but the numbers went the other direction, the coordinates produced by functions g and h will be related by the equation

$$h(P) = -g(P)$$

In this equation, the letter P represents an arbitrary point on line L.

Conclusion

It is helpful to reiterate what we have done. Given points A and B on a line, we wanted a ruler placed so that the resulting coordinate function has two properties:

- the coordinate of point A is zero
- the coordinate of point B is positive

We saw that using a two step process of Ruler Sliding and Ruler Flipping, a ruler could be placed in the correct position.

Let's again take a moment to briefly look ahead at something found in Section 3.9 (Ruler Placement in Neutral Geometry) . The Theorem 11 (Ruler Placement Theorem) on page 87 says.

> If A and B are distinct points on some line L, then there exists a coordinate function h for line L such that $h(A) = 0$ and $h(B)$ is positive.

We see that Theorem 11 is simply telling us that our abstract coordinate functions will exhibit behavior analogous to the ruler placement behavior of drawn rulers discussed above.

3.8. Ruler Placement in Analytic Geometry

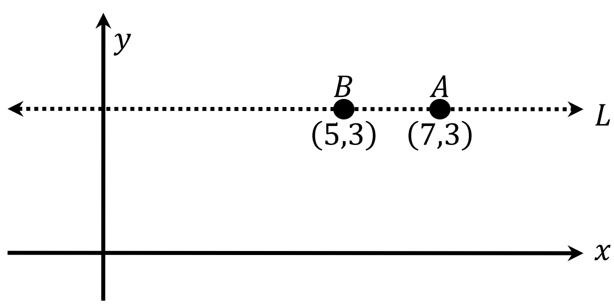
In Section 3.3 The Distance Function and Coordinate Functions in Analytic Geometry (starting on page 65), we saw the introduction of coordinate functions for two lines called L and M. In this section, we will revisit those coordinate functions and obtain additional, new coordinate function for lines L and M by modifying those old coordinate functions. We will see how the modifications of the coordinate functions could be thought of as Ruler Sliding, Ruler Flipping, and Ruler Placement for coordinate functions in Analytic Geometry.

3.8.1. Examples involving the line *L*.

Our first example in Section 3.3 The Distance Function and Coordinate Functions in Analytic Geometry (starting on page 65) involved the line L defined as follows.

$$L = \{(x, y) \text{ such that } y = 3\}$$

This is a horizontal line, parallel to the *x*-axis but 3 units above the *x*-axis. Points $A = (7,3)$ and $B = (5,3)$ are on line L.



In this section, we will be interested in finding a coordinate function for line L that has the following two properties:

- The coordinate of A is zero.
- The coordinate of B is positive.

In Section 3.3 we saw the introduction of a coordinate function for line L. The coordinate function was the function $f: L \to \mathbb{R}$ defined by the equation

$$f(x, y) = x$$

Recall that using the function f, we obtain the following coordinates for points A and B.

- The coordinate of A is the real number $f(A) = f(7,3) = 7$.

- The coordinate of B is the real number $f(B) = f(5,3) = 5$.

We see that using the coordinate function f,

- The coordinate of A is not zero.
- The coordinate of B is positive.

So coordinate function f *does not* have the two properties that we want. But it turns out that we can modify coordinate function f to obtain a new coordinate function that *does* have the two properties that we want. The modification will be done in two steps.

Modification Step 1: Ruler Sliding

Define a new function $g: L \to \mathbb{R}$ is by the equation

$$g(x,y) = f(x,y) - f(A) = f(x,y) - 7 = x - 7$$

Notice that the function g is obtained by adding a constant -7 to the function f.

Using points A and B as inputs to the function g, we obtain the following outputs.

- When point A is the input, the output is the real number $g(A) = g(7,3) = 0$.
- When point B is the input, the output is the real number $g(B) = g(5,3) = -2$.

Also observe that

$$|g(A) - g(B)| = |0 - (-2)| = |2| = 2$$

so the equation

$$|g(A) - g(B)| = d(A,B)$$

is satisfied. The fact that this equation is satified is what qualifies g to be called a *coordinate function* for line L. We can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function g, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $g(A) = g(7,3) = 0$.
- The coordinate of point B is the real number $g(B) = g(5,3) = -2$.

Observe that using coordinate function g,

- The coordinate of A is zero.
- The coordinate of B is negative.

So coordinate function g has the first of the two properties that we want, but it does not have the second property.

Realize that the mathematical operation of adding a constant -7 to the function f corresponds to sliding the ruler for f seven units along the line L. The resulting new ruler--the new coordinate function--is named g. So we see that the operation of sliding a ruler along a line in a drawing has an analog in the world of Analytic Geometry: it corresponds to adding a constant to a coordinate function to get a new coordinate function.

Modification Step 2: Ruler Flipping

Define a new function $h: L \to \mathbb{R}$ by the equation

$$h(x,y) = -g(x,y) = -\big(f(x,y) - f(A)\big) = -(f(x,y) - 7) = 7 - f(x,y) = 7 - x$$

Notice that the function h is obtained by multiplying the function g by -1.

Using points A and B as inputs to the function h, we obtain the following outputs.

- When point A is the input, the output is the real number $h(A) = h(7,3) = 0$.
- When point B is the input, the output is the real number $h(B) = h(5,3) = 2$.

Also observe that

$$|h(A) - h(B)| = |0 - 2| = |-2| = 2$$

so the equation

$$|h(A) - h(B)| = d(A,B)$$

is satisfied. The fact that this equation is satified is what qualifies h to be called a *coordinate function* for line L. We can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function h, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $h(A) = h(7,3) = 0$.
- The coordinate of point B is the real number $h(B) = h(5,3) = 2$.

Observe that using coordinate function h,

- The coordinate of A is zero.
- The coordinate of B is positive.

So coordinate function h has both of the two properties that we want.

Realize that the mathematical operation of multiplying the function g by -1 corresponds to flipping the ruler for g. The resulting new ruler--the new coordinate function--is named h. So we see that the operation of flipping a ruler in a drawing has an analog in the world of Analytic Geometry: it corresponds to multiplying a coordinate function by -1 to get a new coordinate function.

Conclusion: Ruler Placement in Analytic Geometry

In this subsection, we started with a line L. We wanted a coordinate function for line L that would have the following two properties:

- The coordinate of A is zero.
- The coordinate of B is positive.

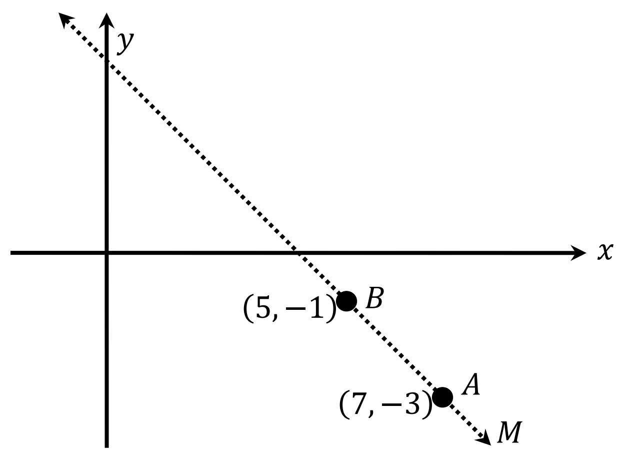
We started with a coordinate function f that did not have these properties. Then we modified f in two steps. In Step 1, we obtained a new coordinated function g by adding a real number constant to f. This was analogous to Ruler Sliding in a drawing. In Step 2, we obtained a new coordinate function h by multiplying g by the number -1. This was analogous to Ruler Flipping in a drawing. We observed that the function h had both of the properties that we wanted. So the two step modification was analagous to Ruler Placement in a drawing.

3.8.2. Examples involving the line M.

Our second example in Section 3.3 The Distance Function and Coordinate Functions in Analytic Geometry (starting on page 65) involved the slanting line M defined as follows.

$$M = \{(x, y) \text{ } such \text{ } that \text{ } x + y = 4\}$$

That is, $y = -x + 4$. Points $A = (7, -3)$ and $B = (5, -1)$ are on on line M.



As we did above for line L, we will be interested in finding a coordinate function for line M that has the following two properties:

- The coordinate of A is zero.
- The coordinate of B is positive.

In Section 3.3 we saw the introduction of a coordinate function for line M. The coordinate function was the function $f: M \to \mathbb{R}$ defined by the equation

$$f(x, y) = x\sqrt{2}$$

Recall that using the function f, we obtain the following coordinates for points A and B.

- The coordinate of A is the real number $f(A) = f(7,3) = 7\sqrt{2}$.
- The coordinate of B is the real number $f(B) = f(5,3) = 5\sqrt{2}$.

We see that using the coordinate function f,

- The coordinate of A is not zero.
- The coordinate of B is positive.

So coordinate function f *does not* have the two properties that we want. But again, it turns out that we can modify coordinate function f to obtain a new coordinate function that *does* have the two properties that we want. Again, the modification will be done in two steps.

Modification Step 1: Ruler Sliding

Define a new function $g: M \to \mathbb{R}$ is by the equation

$$g(x,y) = f(x,y) - f(A) = f(x,y) - 7\sqrt{2} = x\sqrt{2} - 7\sqrt{2}$$

Notice that the function g is obtained by adding a constant $-7\sqrt{2}$ to the function f.

Using points A and B as inputs to the function g, we obtain the following outputs.

- When point A is the input, the output is the real number $g(A) = g(7,3) = 0$.
- When point B is the input, the output is the real number $g(B) = g(5,3) = -2\sqrt{2}$.

Also observe that

$$|g(A) - g(B)| = \left|0 - \left(-2\sqrt{2}\right)\right| = \left|2\sqrt{2}\right| = 2\sqrt{2}$$

so the equation

$$|g(A) - g(B)| = d(A,B)$$

is satisfied. The fact that this equation is satified is what qualifies g to be called a *coordinate function* for line M. We can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function g, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $g(A) = g(7,3) = 0$.
- The coordinate of point B is the real number $g(B) = g(5,3) = -2\sqrt{2}$.

Observe that using coordinate function g,

- The coordinate of A is zero.
- The coordinate of B is negative.

So coordinate function g has the first of the two properties that we want, but it does not have the second property.

Realize that the mathematical operation of adding a constant $-7\sqrt{2}$ to the function f corresponds to sliding the ruler for f a distance of $7\sqrt{2}$ units along the line M. The resulting new ruler--the new coordinate function--is named g. Again we see that the operation of sliding a ruler along a line in a drawing has an analog in the world of Analytic Geometry: it corresponds to adding a constant to a coordinate function to get a new coordinate function.

Modification Step 2: Ruler Flipping

Define a new function $h: M \to \mathbb{R}$ by the equation

$$\begin{aligned} h(x,y) &= -g(x,y) \\ &= -\big(f(x,y) - f(A)\big) \\ &= -\big(f(x,y) - 7\sqrt{2}\big) \\ &= 7\sqrt{2} - f(x,y) \\ &= 7\sqrt{2} - x\sqrt{2} \end{aligned}$$

Notice that the function h is obtained by multiplying the function g by -1.

Using points A and B as inputs to the function h, we obtain the following outputs.

- When point A is the input, the output is the real number $h(A) = h(7,3) = 0$.
- When point B is the input, the output is the real number $h(B) = h(5,3) = 2\sqrt{2}$.

Also observe that

$$|h(A) - h(B)| = |0 - 2\sqrt{2}| = |-2\sqrt{2}| = 2\sqrt{2}$$

so the equation

$$|h(A) - h(B)| = d(A,B)$$

is satisfied. The fact that this equation is satified is what qualifies h to be called a *coordinate function* for line L. We can rewrite the two sentences above about inputs and outputs, using instead the terminology of coordinate functions and coordinates. We can say that using the coordinate function h, we obtain the following coordinates for points A and B.

- The coordinate of point A is the real number $h(A) = h(7,3) = 0$.
- The coordinate of point B is the real number $h(B) = h(5,3) = 2\sqrt{2}$.

Observe that using coordinate function h,

- The coordinate of A is zero.
- The coordinate of B is positive.

So coordinate function h has both of the two properties that we want.

Realize that the mathematical operation of multiplying the function g by -1 corresponds to flipping the ruler for g. The resulting new ruler--the new coordinate function--is named h. Again we see that the operation of flipping a ruler in a drawing has an analog in the world of Analytic Geometry: it corresponds to multiplying a coordinate function by -1 to get a new coordinate function.

Conclusion: Ruler Placement in Analytic Geometry

In this subsection we, we started with a line M. We wanted a coordinate function for line M that would have the following two properties:

- The coordinate of A is zero.
- The coordinate of B is positive.

We started with a coordinate function f that did not have these properties. Then we modified f in two steps. In Step 1, we obtained a new coordinated function g by adding a real number constant to f. This was analogous to Ruler Sliding in a drawing. In Step 2, we obtained a new coordinate function h by multiplying g by the number -1. This was analogous to Ruler Flipping in a drawing. We observed that the function h had both of the properties that we wanted. So the two step modification was analagous to Ruler Placement in a drawing.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 3.11 on page 94.

3.9. Ruler Placement in Neutral Geometry

Having studied Ruler Sliding, Ruler Flipping, and Ruler Placement in both drawings and Analytic Geometry, we are ready to see how the abstract version of each of those three behaviors is manifest in Neutral Geometry. Notice that the Neutral Geometry axioms do not say anything about Ruler Sliding, Ruler Flipping, or Ruler Placement. We will have to prove in theorems that the three behaviors do, in fact, occur in Neutral Geometry.

Here is the first theorem about ruler behavior in Neutral Geometry. It has to do with Ruler Sliding and Ruler Flipping.

Theorem 10 (Ruler Sliding and Ruler Flipping) Lemma about obtaining a new coordinate function from a given one

Suppose that $f: L \to \mathbb{R}$ is a coordinate function for a line L.

(A) (Ruler Sliding) If c is a real number constant and g is the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) + c$, then g is also a coordinate function for line L.

(B) (Ruler Flipping) If g is the function $g: L \to \mathbb{R}$ defined by $g(P) = -f(P)$, then g is also a coordinate function for line L.

A digression to discuss proof structure

Before embarking on the proof, it is worthwhile to discuss the proof structure. We will consider the structure of the proof of Statement (A) and then I will present the proof. A similar structure will be needed for the proof of Statement (B), which you will prove as an exercise.

Since Statement (A) is a conditional statement, step (1) of the proof will of course be simply a statement of the hypothesis of Statement (A), and the final step of the proof will be a statement of the conclusion of Statement (A), with some justification provided. Look ahead to the proof provided below and confirm that it does indeed begin and end in this way.

Now, working backwards from the end of the proof, let's think about how the final statement of the proof will need to be justified. The final statement of the proof says that some function g is a coordinate function for some line L. But *coordinate function* is a defined term. The only way to prove that some function is a coordinate function is to prove that the function does indeed have all of the characteristics described in the definition of coordinate function. For reference, here is part of that definition.

Definition 21: Coordinate Function (page 70)
Words: f is a coordinate function on line L.
Meaning: f is a function with domain L and codomain $\mathbb{R}$ (that is, $f: L \to \mathbb{R}$) that has the following properties:
(1) f is a one-to-one correspondence. That is, f is both one-to-one and onto.
(2) f "agrees with" the distance function d in the following way:
For all points P,Q on line L, the equation $|f(P) - f(Q)| = d(P, Q)$ is true.

If we want to prove that g is a coordinate function for line L, we will need to prove that g does have all the characteristics listed above. That means we will need to prove four things:

1. **Prove that g is a function with domain L and codomain $\mathbb{R}$ (that is, $g: L \to \mathbb{R}$).** Notice that $g: L \to \mathbb{R}$ is already stated as part of the description of g in Statement (A), as if it were already known to be a fact. How is it already known? Well, the proof of the fact is so easy that once you see the proof in this remark, you will understand why the proof of the fact is omitted in the real proof.

   To prove that $g: L \to \mathbb{R}$ is true, we would need to prove that g will accept any given point on line L as input, and will produce exactly one real number as output. So suppose that P is some given point on line L. Then $f(P)$ is a real number, by the fact that $f: L \to \mathbb{R}$ is true since f is known to be a coordinate function. Then $f(P) + c$ is also a real number. But $g(P) = f(P) + c$. That means that $g(P)$ does in fact represent a real number. So $g: L \to \mathbb{R}$ is true.

   So the proof that $g: L \to \mathbb{R}$ is so easy that the fact is stated without proof, as part of Statement (A).

2. **Prove that g is one-to-one.** Recall that to say that a function is one-to-one means that if two inputs cause two outputs that are equal, then the two inputs must have also been equal as well. In our situation, we must prove that if $g(P) = g(Q)$, then $P = Q$.

3. **Prove that g is onto.** Recall that to say that a function is onto means that for any desired output in the codomain, there exists an input in the domain that will cause the function to give that desired output. In our situation, we must prove that for any real number y, there exists some point P on line L such that$g(P) = y$.

4. **Prove that g "agrees with" the distance function d in the following way:**
   For all points P, Q on line L, the equation $|g(P) - g(Q)| = d(P, Q)$ is true.

If you look ahead to the proof of Statement (A), you will see that the proof is organized into three sections, corresponding to items 2, 3, and 4 on the above list.

End of digression

Here, then, is the proof of Theorem 10 Statement (A):

Proof that Statement (A) is true

(1) Suppose that $f: L \to \mathbb{R}$ is a coordinate function for a line L. And suppose that c is a real number constant, and that g is the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) + c$.

Prove that the function g is one-to-one.

(2) Suppose that P, Q are points on line L such that $g(P) = g(Q)$.
(3) Then $f(P) + c = f(Q) + c$. (by definition of how function g works.)
(4) Then $f(P) = f(Q)$. (by arithmetic)
(5) Then $P = Q$. (by statement 4 and the fact that function f is known to be one-to-one because f is known to be a coordinate function.)
(6) Conclude that function g is one-to-one. (by steps (2), (5) and the definition of one-to-one)

Prove that the function g is onto.

(7) Suppose that a real number y is given. (This is our desired output from the function g.)
(8) Observe that $y - c$ is also a real number. Consider this new real number as a desired output for the function f. Because function f is known to be a coordinate function, we know that f is onto. That is, there exists an input in the domain that will cause the function f to give the desired output of $y - c$. In other words, there exists some point P on line L such that $f(P) = y - c$.
(9) Now consider what happens when we use the point P as input to the function g. The corresponding output is $g(P) = f(P) - c = (y - c) + c = y$. In other words, there exists some point P on line L such that $g(P) = y$.
(10) Conclude that function g is onto. (by steps (7), (9) and the definition of onto)

Prove that the function g "agrees with" the distance function d.

(11) For any two points P and Q on line L, we have

$$\begin{aligned} |g(P) - g(Q)| &= |(f(P) + c) - (f(Q) + c)| \\ &= |f(P) - f(Q)| \\ &= d(P, Q) \end{aligned}$$

Conclusion

(12) Conclude that g is a coordinate function for line L. (by steps (6) (10), (11), and the definition of Coordinate Function (Definition 21).

End of proof that Statement (A) is true

As mentioned above, you will be asked to prove Theorem 10 Statement (B) as an exercise.

Here is the second theorem about ruler behavior in Neutral Geometry. It has to do with Ruler Placement.

Theorem 11 Ruler Placement Theorem

If A and B are distinct points on some line L, then there exists a coordinate function h for line L such that $h(A) = 0$ and $h(B)$ is positive.

Proof

Part 1: Find a coordinate function with the correct behavior at A.

(1) Suppose that A and B are distinct points on some line L.
(2) There exists a coordinate function f for line L. **(Justify.)**
(3) Let c be the real number defined by $c = f(A)$. That is, c is the real number coordinate of point A using coordinate function f on line L.
(4) Let g be the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) - c$.
(5) The function g is also a coordinate function for line L. **(Justify.)**
(6) Observe that $g(A) = f(A) - c = f(A) - f(A) = 0$.

Part 2: Find a coordinate function with the correct behavior at both A and B.

(7) We know that $g(B) \neq 0$ **(Justify.)**. Therefore, $g(B)$ will be either positive or negative.

Case 1: Suppose that $g(B)$ is positive.

(8) If $g(B)$ is positive, then we can just let h be coordinate function g. Then h is a coordinate function for line L such that $h(A) = 0$ and $h(B)$ is positive.

Case 2: Suppose that $g(B)$ is negative.

(9) If $g(B)$ is negative, then let h be the function $h: L \to \mathbb{R}$ defined by $h(P) = -g(P)$.
(10) The function h is also a coordinate function for line L. **(Justify.)**
(11) Observe that $h(A) = -g(A) = -0 = 0$.
(12) Also observe that $h(B) = -g(B) = -negative = positive$.
(13) So h is a coordinate function for line L such that $h(A) = 0$ and $h(B)$ is positive.

Conclusion of Cases

(14) We see that in either case, it is possible to define a coordinate function h for line L such that $h(A) = 0$ and $h(B)$ is positive.

End of proof

Here, it is worthwhile to pause for a moment and consider the significance of the proofs of Theorem 10 and Theorem 11.

Back in Section 3.7 (Ruler Placement in Drawings, which began on page 75), the notions of Ruler Sliding, Ruler Flipping, and Ruler Placement in drawings were very concrete. They are actual things that you do with real rulers. Then in Section 3.8 (Ruler Placement in Analytic Geometry, which began on page 81), you saw analogous notions of Ruler Sliding, Ruler Flipping, and Ruler Placement in Analytic Geometry. You worked with actual equations, and the distance function and coordinate functions were given by actual formulas, whose inputs involved numbers and whose outputs were numbers. Ruler Sliding, Ruler Flipping, and Ruler Placement were described by actual formulas.

Now you have seen that there are notions of Ruler Sliding, Ruler Flipping, and Ruler Placement in Neutral Geometry, as well. But in Neutral Geometry, the distance function is not given by a formula, and coordinate functions are not given by a formula. Indeed, there is no way that they could be given by formulas, because the inputs to the functions are points, which in Neutral Geometry are *undefined objects*. So in the proofs of Theorem 10 and Theorem 11, the distance function d and the coordinate functions f, g, h are abstract, not given by formulas. The notions of Ruler Sliding, Ruler Flipping, and Ruler Placement are described in terms of actual mathematical operations done to these abstract functions. The theorems were proved by referring to the definition of an abstract coordinate function and by careful use of the definitions of one-to-one functions and onto functions.

3.10. Distance and Rulers in High School Geometry Books

Many High School Geometry books use a set of axioms called the SMSG (School Mathematics Study Group) axioms. Here are the first four SMSG axioms. (The word *postulate* is used instead of the word axiom.)

SMSG Postulate 1: Given any two distinct points there is exactly one line that contains them.
SMSG Postulate 2: (Distance Postulate) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.
SMSG Postulate 3: (Ruler Postulate) The points of a line can be placed in a correspondence with the real numbers such that:

- To every point of the line there corresponds exactly one real number.
- To every real number there corresponds exactly one point of the line.
- The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

SMSG Postulate 4: (Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Distance in the SMSG Axioms

Compare the SMSG Postulate 2 (the Distance Postulate) to our Neutral Geomtry axiom about distance:

<N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

There is one obvious difference in the two axioms: Our distance axiom <N4> uses the terminology of functions, while the SMSG distance postulate does not. Why not? Well, the terminology of functions, and the symbols used as an abbreviation for that terminology, are something that is typically taught in a 2^{nd}- or 3^{rd}-year college course. So a high school book cannot use notation such as $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$.

But there is another, more important, difference between the two axioms. The SMSG postulates only talk about the distance between two points that are known to be distinct, while our Neutral Geometry Axioms talk about the distance between any two points, not necessarily distinct. If you consider what our neutral geometry axioms tell us about how our neutral geometry distance function d behaves, you will see that if points P and Q are actually the same point, then $d(P, Q) = 0$. (Can you explain why?) But if you consider what the SMSG postulates tell us about distance between points in their geometry, if points P and Q are actually the same point, then the distance $d(P, Q)$ is undefined.

You might think that this discussion is silly, that everybody knows that if points P and Q are actually the same point, then $d(P,Q) = 0$, even if the SMSG axioms don't say so. But the point is we *don't* know it unless the axioms say that it is true (or we prove it in a theorem). So in our Neutral Geometry, we can say that if points P and Q are actually the same point, then $d(P,Q) = 0$, while in the SMSG geometry, we cannot say that.

You might wonder if it really matters whether or not our distance function is capable of measuring the distance $d(P,Q)$ if points P and Q are actually the same point. Well, it is indeed useful to be able to do that. And in the definition of a distance function conventionally used in mathematics, points P and Q are allowed to be the same point.

One more note: Notice that in the distance formula from Analytic Geometry (the formula involving the square root), points P and Q can be the same point. If they are, then $d(P,Q) = 0$.

Coordinates in the SMSG Axioms

Compare what the SMSG Axioms say about coordinates to what our Neutral Geometry Axioms and Definitions say about coordinates:

From the SMSG Axioms:

SMSG Postulate 3: (Ruler Postulate) The points of a line can be placed in a correspondence with the real numbers such that:

- To every point of the line there corresponds exactly one real number.
- To every real number there corresponds exactly one point of the line.
- The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

From our Neutral Geometry Axioms and Definitions, we have a definition and an axiom:

Definition 21 Coordinate Function

Words: f is a coordinate function on line L.

Meaning: f is a function with domain L and codomain $\mathbb{R}$ (that is, $f: L \to \mathbb{R}$) that has the following properties:

(1) f is a one-to-one correspondence. That is, f is both one-to-one and onto.

(2) f "agrees with" the distance function d in the following way:

For all points P,Q on line L, the equation $|f(P) - f(Q)| = d(P,Q)$ is true.

Additional Terminology: In standard function notation, the symbol $f(P)$ denotes the output of the coordinate function f when the point P is used as input. Note that $f(P)$ is a real number. The number $f(P)$ is called *the coordinate of point P on line L.*

Additional Notation: Because a coordinate function is tied to a particular line, it might be a good idea to have a notation for the coordinate function that indicates which line the coordinate function is tied to. We could write f_L for a coordinate function on line L. With that notation, the symbol $f_L(P)$ would denote the coordinate of point P on line L. But although it might be clearer, we do not use the symbol f_L. We just use the symbol f.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Notice that the main difference is that our Neutral Geometry uses the terminology and notation of functions, while the SMSG axioms do not. But again notice that when the SMSG Postulate describes the relationship between the coordinates of two points on a line and the distance between those two points, the points are required to be *two distinct points.* No such requirement is made in our Neutral Geometry.

Ruler Placement in the SMSG Axioms

Finally, notice that Ruler Placement behavior is guaranteed in both the SMSG geometry and in our Neutral Geometry, but in different ways:

From the SMSG Geometry:

> **SMSG Postulate 4:** (Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

From our Neutral Geometry:

> **Theorem 11** Ruler Placement Theorem
> If A and B are distinct points on some line L, then there exists a coordinate function h for line L such that $h(A) = 0$ and $h(B)$ is positive.

You see that the SMSG Axioms include a Postulate that guarantees Ruler Placement behavior, while in our Neutral Geometry, Ruler Placement behavior is proven as a theorem.

Remember that if an axiom that can be proven true (or false) as a consequence of the other axioms, then that axiom is *not* an independent axiom. We see that the SMSG Axioms includes an axiom (SMSG Postulate 4, the Ruler Placement Postulate) that is not independent. So the set of SMSG Axioms is not an independent axiom system.

An obvious question is, why include an axiom that is not independent? Why not just leave it off the axiom list and prove it as a theorem, as we have?

For an answer to that question, consider what was involved in our proof of Neutral Geometry Theorem 11, the Ruler Placement Theorem. In Section 3.9 (Ruler Placement in Neutral Geometry, which starts on page 87), we first proved Theorem 10 (Ruler Sliding and Ruler Flipping). The statement of that theorem used function terminology and notation. Remember that high school students don't know that terminology. If we were to try to reword the statement of the theorem for a high school audience, without using function terminology and notation, we would find that the statement would become much more cumbersome. And the proof of the theorem is densly-written, using both function notation and absolute value notation. That proof is simply above the level of a high school course. Similarly, our proof of Neutral Geometry Theorem 11, the Ruler Placement Theorem, also made extensive use of function terminology and notation. Again, the proof is above the level of a high school course.

So our list of ten Neutral Geometry axioms is shorter than the list of SMSG axioms, and our list of axioms is independent. But a trade-off is that in our course, we have to prove some difficult theorems.

3.11. Exercises for Chapter 3

Exercises for Section 3.1 Neutral Geometry Axioms and First Six Theorems

[1] How are points defined in Neutral Geometry?

[2] How are lines defined in Neutral Geometry?

[3] In Neutral Geometry, what does it mean to say that "the point lies on the line"?

[4] After graduation, you land a job hosting the show "Geometry Today" on National Public Radio. Somebody calls in to complain about Axiom <N2>. He says that lines in Neutral Geometry should not be limited to containing only two points, because clearly lines in drawings contain more than two points. How do you respond?

[5] What do the Neutral Geometry axioms say about parallel lines?

[6] Prove Neutral Geometry Theorem 1 (In Neutral Geometry, if L and M are distinct lines that intersect, then they intersect in only one point.), which was presented on page 61.

[7] Prove Neutral GeometryTheorem 2 (In Neutral Geometry, there exist three non-collinear points.), which was presented on page 61.

[8] Prove Neutral GeometryTheorem 3 (In Neutral Geometry, there exist three lines that are not concurrent.), which was presented on page 61.

[9] Prove Neutral GeometryTheorem 4 (In Neutral Geometry, for every point P, there exists a line that does not pass through P.), which was presented on page 61.

[10] Prove Neutral GeometryTheorem 5 (In Neutral Geometry, for every point P, there exist at least two lines that pass through P.), which was presented on page 61.

Exercises for Section 3.3 The Distance Function and Coordinate Functions in Analytic Geometry

[11] Let L be the line described as the set $L = \{(x, y) \text{ } such \text{ } that \text{ } y = x + 3\}$. Let A and B be the two points $A = (7,10)$ and $B = (5,8)$. Both points are on line L.

(A) Find $d(A, B)$.

(B) Define the function $f: L \to \mathbb{R}$ by the equation $f(x, y) = x$.

(i) Find $f(A)$.

(ii) Find $f(B)$.

(iii) Find $|f(A) - f(B)|$.

(iv) Is the equation $|f(A) - f(B)| = d(A, B)$ true?

(v) Could f be a coordinate function for line L? Explain.

(C) Define the function $g: L \rightarrow \mathbb{R}$ by the equation $g(x,y) = x\sqrt{2}$.
(i) Find $g(A)$.
(ii) Find $g(B)$.
(iii) Find $|g(A) - g(B)|$.
(iv) Is the equation $|g(A) - g(B)| = d(A,B)$ true?
(v) Could g be a coordinate function for line L? Explain.

Exercises for Section 3.4 The Distance Function and Coordinate Functions in Neutral Geometry

[12] In Section 3.3, The Distance Function and Coordinate Functions in Analytic Geometry, which started on page 65, we discussed the familiar distance function

$$d\big((x_1,y_1),(x_2,y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

That was familiar and not too hard. Why can't we just define distance that way in Neutral Geometry? Why were we never given any formula for the Neutral Geometry distance function?

[13] Prove Neutral GeometryTheorem 7 (about how many points are on lines in Neutral Geometry), which was presented on page 70.

Exercises for Section 3.6 Two Basic Properties of the Distance Function in Neutral Geometry

[14] Justify the steps in the proof of Theorem 9 (The Distance Function on the Set of Points, the function d, is Symmetric.), which was presented on page 74.

Exercises for Section 3.8 Ruler Placement in Analytic Geometry

[15] Let L be the line described as the set $L = \{(x,y) \text{ such that } y = x + 3\}$. Let A and B be the two points $A = (7,10)$ and $B = (5,8)$. Both points are on line L. Find a coordinate function h for line L such that when using the coordinate function h,

- The coordinate of A is zero. That is, $h(A) = 0$.
- The coordinate of B is positive. That is, $h(B)$ is positive.

[16] Let M be the vertical line described as the set $M = \{(x,y) \text{ such that } x = 3\}$. Let A and B be the two points $A = (3,10)$ and $B = (3,8)$. Both points are on line M. Find a coordinate function h for line M such that when using the coordinate function h,

- The coordinate of A is zero. That is, $h(A) = 0$.
- The coordinate of B is positive. That is, $h(B)$ is positive.

Exercises for Section 3.9 Ruler Placement in Neutral Geometry

[17] Justify the steps in the proof of Theorem 10A (Ruler Sliding), which was presented on page 87.

[18] Prove Theorem 10B (Ruler Flipping), which was presented on page 87.

[19] Justify the steps in the proof of Theorem 11 (Ruler Placement Theorem), which was presented on page 89.

Exercises for Section 3.10 Distance and Rulers in High School Geometry Books

[20] In the reading, we discussed that the SMSG Axioms do not use function terminology or function notation, and that there is at least one SMSG Axiom (SMSG Postulate 4, the Ruler Placement Postulate) the appears in our Neutral Geometry as a theorem (our Neutral Geometry Theorem 11, the Ruler Placement Theorem). Consider our Neutral Geometry Theorem 10:

> **Theorem 10:** (Ruler Sliding and Ruler Flipping) Lemma about obtaining a new coordinate function from a given one
>
> Suppose that $f: L \to \mathbb{R}$ is a coordinate function for a line L.
>
> (A) (Ruler Sliding) If c is a real number constant, and g is the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) + c$, then g is also a coordinate function for line L.
>
> (B) (Ruler Flipping) If g is the function $g: L \to \mathbb{R}$ defined by $g(P) = -f(P)$, then g is also a coordinate function for line L.

Suppose that you wanted to include the statement of our Neutral Geometry Theorem 10 as a Postulate to be included in the list of SMSG Axioms, to be used by high school students. Try rewriting the statement Neutral Geometry Theorem 10 in the style of the other SMSG postulates. That is, rewrite it without using function terminology or function notation.

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4. Neutral Geometry II: More about the Axioms of Incidence and Distance

4.1. Betweenness

In the previous chapter, we focused on understanding the *idea* of the distance function and coordinate functions. The discussion and the theorems were mostly about examining the behavior of the distance function and coordinate functions (or rulers) in drawings and in Analytic Geometry, and then proving (in Theorem 10 (Ruler Sliding and Ruler Flipping) and Theorem 11 (Ruler Placement Theorem)) that the abstract distance function and abstract coordinate functions in Neutral Geometry behave in the same way.

Now we turn our attention to another concept from the world of drawings: the notion of *betweenness*. We will see that there is an analogous notion of *betweenness* in Neutral Geometry.

The notion of *betweenness* is an important one in geometry. Given three distinct points lying on a line in a drawing, we are all familiar with the simple idea that one of the points is "between" the other two. We expect that our axiomatic geometry will have the same behavior. But remember that if we want our axiomatic geometry to have a certain behavior, we need to specify that behavior in the axioms, or prove in a theorem that the axiom system does have the behavior. In our axiom system, it turns out that we do not need to include any axioms specifically about *betweenness*. Rather, we can define precisely what we mean by *betweenness*, and then prove in theorems that our abstract geometry will in fact have the *betweenness* behavior that we expect.

Many of the theorems in this chapter have proofs that are rather tedious and involve a level of detail slightly above what is really needed for a Junior-level course in Axiomatic Geometry. The proofs are included for readers interested in advanced topics and for graduate students.

4.1.1. Betweenness of Real Numbers

We start by defining betweenness for real numbers and observing some facts about it. We do this for two reasons: (1) it introduces us to some of the terminology and symbols commonly used in discussions about betweenness and (2) it will turn out that the definition betweenness of points and theorems about betweenness of points will make use of the facts of betweenness for real numbers. Here is our first definition.

Definition 23 betweenness for real numbers
words: "y is between x and z", where x, y, and z are real numbers.
symbol: $x * y * z$, where x, y, and z are real numbers
meaning: $x < y < z$ or $z < y < x$.
additional symbol: the symbol $w * x * y * z$ means $w < x < y < z$ or $z < y < x < w$, etc.

The following theorem states some facts about betweenness for real numbers. Its claims are simple and mostly follow from the definition of betweenness and from the axioms for the real numbers. In this book, we won't bother proving things that are really just consequences of the axioms for the real numbers. So we will accept this theorem without proof.

Theorem 12 facts about betweenness for real numbers

(A) If $x * y * z$ then $z * y * x$.

(B) If x, y, z are three distinct real numbers, then exactly one is between the other two.

(C) Any four distinct real numbers can be named in an order w, x, y, z so that $w * x * y * z$.

(D) If a and b are distinct real numbers, then

(D.1) There exists a real number c such that $a * c * b$.

(D.2) There exists a real number d such that $a * b * d$

The fact about real numbers proven in the next theorem does not play much of a role in computations involving real numbers. But the fact is useful in a future proof, so we state and prove the fact here. In other words, the following theorem could be called a Lemma. The proof is included for readers interested in advanced topics, and for graduate students.

Theorem 13 Betweenness of real numbers is related to the distances between them.

Claim: For distinct real numbers x, y, z, the following are equivalent

(A) $x * y * z$

(B) $|x - z| = |x - y| + |y - z|$. That is, $d_{\mathbb{R}}(x,z) = d_{\mathbb{R}}(x,y) + d_{\mathbb{R}}(y,z)$.

Proof (for readers interested in advanced topics and for graduate students)

Part I: Show that (A) ➔ (B)

(1) Suppose that real numbers x, y, z have the property $x * y * z$.

(2) Either $x < y < z$ or $z < y < x$.

Case 1: $z < y < x$

(3) If $z < y < x$, then

$$\begin{aligned} |x - z| &= x - z \ (because\ x - z\ is\ positive) \\ &= x - y + y - z \ (trick) \\ &= |x - y| + |y - z| \ (because\ x - y\ is\ positive\ and\ y - z\ is\ positive) \end{aligned}$$

Case 2: $x < y < z$

(4) If $x < y < z$, then

$$\begin{aligned} |x - z| &= |z - x| \\ &= z - x \ (because\ z - x\ is\ positive) \\ &= z - y + y - x \ (trick) \\ &= |z - y| + |y - x| \ (because\ z - y\ is\ positive\ and\ y - x\ is\ positive) \\ &= |x - y| + |y - z| \end{aligned}$$

Conclusion of cases

(5) We see that in either case, $|x - z| = |x - y| + |y - z|$ is true, so statement (2) is true.

End of proof of Part I.

Part II: Show that ~(A) ➔ ~(B)

(1) Suppose that Statement (A) is false. That is, suppose that real numbers x, y, z do not have the property $x * y * z$.

(2) Either $y * x * z$ or $x * z * y$. **(Justify.)**

Case 1: $y * x * z$

(3) If $y * x * z$ then $|y - z| = |y - x| + |x - z|$ by result of Proof Part I with letters changed.

(4) Subtracting, we find that

$$|x - z| = -|y - x| + |y - z|$$

$$= -|x-y| + |y-z|$$
$$= |x-y| + |y-z| - 2|x-y|$$

We see that $|x-z|$ does not equal $|x-y|+|y-z|$ because there is an additional term $-2|x-y|$. (We know that this additional term is nonzero because $x \neq y$.) Therefore, statement (B) is false.

Case 2: $x * z * y$

(5) If $x * z * y$ then $|x-y| = |x-z| + |z-y|$ by result of Proof Part I with letters changed.

(6) Subtracting, we find that

$$\begin{aligned} |x-z| &= |x-y| - |z-y| \\ &= |x-y| - |y-z| \\ &= |x-y| + |y-z| - 2|y-z| \end{aligned}$$

We see that $|x-z|$ does not equal $|x-y|+|y-z|$ because there is an additional term $-2|y-z|$. (We know that this additional term is nonzero because $y \neq z$.) Therefore, statement (B) is false.

Conclusion of Cases

(7) In either case, we see that Statement (B) is false.

End of proof of part II

4.1.2. Betweenness of Points

In the previous section, we established some of the terminology of betweenness for real numbers, and stated some of the facts about the behavior of betweenness for real numbers. In the current section, we will develop the notion of betweenness of points. As mentioned above, we will find that the notion of betweenness of points and theorems about betweenness of points make frequent use of the facts about betweenness of real numbers.

The first theorem of the section is really just a Lemma that will make it possible to define betweenness of points in a simple way. The proof is included for readers interested in advanced topics, and for graduate students.

Theorem 14 Lemma about betweenness of coordinates of three points on a line

If P, Q, R are three distinct points on a line L, and f is a coordinate function on line L, and the betweenness expression $f(P) * f(Q) * f(R)$ is true, then for any coordinate function g on line L, the expression $g(P) * g(Q) * g(R)$ will be true.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that P, Q, R are three distinct points on a line L, and f is a coordinate function on line L, and $f(P) * f(Q) * f(R)$, and that g is a coordinate function on line L.

(2) Observe that

$$\begin{aligned} |g(P) - g(R)| &= d(P,R) \ \ (\boldsymbol{justify}) \\ &= |f(P) - f(R)| \ \ (\boldsymbol{justify}) \\ &= |f(P) - f(Q)| + |f(Q) - f(R)| \ \ (\boldsymbol{justify}) \\ &= d(P,Q) + d(Q,R) \ \ (\boldsymbol{justify}) \\ &= |g(P) - g(Q)| + |g(Q) - g(R)| \ \ (\boldsymbol{justify}) \end{aligned}$$

(3) Therefore, by Theorem 13, we know that $g(P) * g(Q) * g(R)$.

End of proof

We see that for distinct points P, Q, R lying on a line L, the betweenness properties of the real number coordinates of the three points does not depend on which coordinate function we use on line L. Because of this fact, we are able to make the following definition of betweenness for points.

Definition 24 betweenness of points

words: "Q is between P and R", where P, Q, R are points.
symbol: $P * Q * R$, where P, Q, R are points.
meaning: Points P, Q, R are collinear, lying on some line L, and there is a coordinate function f for line L such that the real number coordinate for Q is between the real number coordinates of P and R. That is, $f(P) * f(Q) * f(R)$.
remark: By Theorem 14, we know that it does not matter which coordinate function is used on line L. The betweenness property of the coordinates of the three points will be the same regardless of the coordinate function used.
additional symbol: The symbol $P * Q * R * S$ means $f(P) * f(Q) * f(R) * f(S)$, etc.

In Theorem 12 (facts about betweenness for real numbers) on page 98, we saw a number of obvious properties of betweenness for real numbers. The following theorem states that there are entirely analogous properties of betweenness for points on a line. Because of the way that we defined betweenness for points on a line in terms of betweenness of their real number coordinates, the proof of this theorem will be easy and will make use of the facts about betweenness of real numbers listed in Theorem 12.

Theorem 15 Properties of Betweenness for Points

(A) If $P * Q * R$ then $R * Q * P$.
(B) For any three distinct collinear points, exactly one is between the other two.
(C) Any four distinct collinear points can be named in an order P, Q, R, S such that $P * Q * R * S$.
(D) If P and R are distinct points, then
 (D.1) There exists a point Q such that $P * Q * R$.
 (D.2) There exists a point S such that $P * R * S$.

Proof (for readers interested in advanced topics and for graduate students)
Proof of Statement (A)

(1) Suppose that $P * Q * R$.
(2) Points P, Q, R are collinear, lying on some line L, and there is a coordinate function f for line L such that $f(P) * f(Q) * f(R)$. **(Justify.)**
(3) $f(R) * f(Q) * f(P)$. **(Justify.)**
(4) $R * Q * P$. **(Justify.)**

End of proof of Statement (A)

Proof of Statement (B)

(1) Suppose that P, Q, R are distinct, collinear points, lying on some line L.
(2) There is a coordinate function f for line L. **(Justify.)** This coordinate function gives us real number coordinates $f(P)$ and $f(Q)$ and $f(R)$ for the three points.
(3) The three real numbers $f(P)$ and $f(Q)$ and $f(R)$ are distinct. **(Justify.)**

(4) Exactly one of the three real numbers $f(P)$ and $f(Q)$ and $f(R)$ is between the other two **(Justify.)**
(5) Exactly one of the points P, Q, R lies between the other two. **(Justify.)**
End of proof of Statement (B)

Proof of Statement (C)
(1) Suppose that four distinct, collinear points are given, lying on some line L.
(2) There is a coordinate function f for line L. **(Justify.)** This coordinate function gives us real number coordinates for the four points.
(3) The four real number coordinates are distinct. **(Justify.)**
(4) The four real number coordinates can be named in increasing order, $p < q < r < s$. With this naming, we can say that the four real numbers have the betweenness property $p * q * r * s$.
(5) Let $P = f^{-1}(p)$ and $Q = f^{-1}(q)$ and $R = f^{-1}(r)$ and $S = f^{-1}(s)$.
(6) Then $P * Q * R * S$. **(Justify.)**
End of proof of Statement (C)

The Proof of Statement (D) is left to the reader

In Theorem 13 (Betweenness of real numbers is related to the distances between them.), found on page 98, we saw that betweenness of real numbers is related to the distances between them. The following theorem states an analogous relationship for betweenness of points. Expect that the proof of this theorem will make use of Theorem 13. The proof is included for readers interested in advanced topics, and for graduate students.

Theorem 16 Betweenness of points on a line is related to the distances between them.
Claim: For distinct collinear points P, Q, R, the following are equivalent
(A) $P * Q * R$
(B) $d(P, R) = d(P, Q) + d(Q, R)$.

Proof (for readers interested in advanced topics and for graduate students)
Part I: Show that (A) ➔ (B)
(1) Suppose that Statement (A) is true. That is, suppose that collinear points P, Q, R have the property $P * Q * R$.
(2) The three points P, Q, R lie on some line L, and there is a coordinate function f for line L such that $f(P) * f(Q) * f(R)$. **(Justify.)**
(3) $|f(P) - f(R)| = |f(P) - f(Q)| + |f(Q) - f(R)|$ **(Justify.)**
(4) $d(P, R) = d(P, Q) + d(Q, R)$ **(Justify.)** So Statement (B) is true.
End of proof of Part I

Part II: Show that ~(A) ➔ ~(B): This part of the proof is left to the reader.

4.1.3. A Lemma about Distances Between Three Distinct, Collinear Points

A new fact about the function d will be useful in the proof of Theorem 65 (The Distance Function Triangle Inequality for Neutral Geometry), found in Section 7.4 on page 175. The new useful fact can be proven using only the axioms and theorems that we have studied so far, so we will prove the fact here and return to it when we get to Theorem 65. Because the fact is only used once in the rest of the book, we will call it a Lemma.

To understand the statement of the Lemma, suppose that P, Q, R are distinct collinear points. If $P * Q * R$, then Theorem 16 tells us that $d(P,R) = d(P,Q) + d(Q,R)$. What if $P * Q * R$ is not true? The answer is the statement of the Lemma. The proof is included for readers interested in advanced topics and for graduate students.

Theorem 17 Lemma about distances between three distinct, collinear points.
If P, Q, R are distinct collinear points such that $P * Q * R$ is not true, then the inequality $d(P,R) < d(P,Q) + d(Q,R)$ is true.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that P, Q, R are distinct, collinear points and that $P * Q * R$ is not true.
(2) There are two possibilities: Either $Q * P * R$ is true or $P * R * Q$ is true. (by Theorem 15)

Case (i) $\boldsymbol{Q * P * R}$ is true.

(3) If $Q * P * R$ is true, then Theorem 16 tells us that $d(Q,R) = d(Q,P) + d(P,R)$. In this case, we can subtract to obtain the equation $d(P,R) = d(Q,R) - d(Q,P)$. We can manipulate this equation in the following way:

$$\begin{aligned} d(P,R) &= d(Q,R) - d(Q,P) \\ &= -d(P,Q) + d(Q,R) \\ &= \ d(P,Q) + d(Q,R) - 2d(P,Q) \\ &< d(P,Q) + d(Q,R) \quad (because\ 2d(P,Q)\ is\ positive) \end{aligned}$$

Case (ii) $\boldsymbol{P * R * Q}$ is true.

(4) If $P * R * Q$ is true then an argument like the one in Step (3) would show that the inequality $d(P,R) < d(P,Q) + d(Q,R)$ is true.

Conclusion of Cases

(5)We see that in either case, the inequality $d(P,R) < d(P,Q) + d(Q,R)$ is true.

End of Proof

4.1.4. Conclusion

In this section, we discussed the notion of betweenness for real numbers. We found that collinear points have analogous betweenness behavior. The existence of coordinate functions for lines was the underlying reason for the analogous behavior. In the next section, we will find that coordinate functions can play a key role in definitions of some new geometric objects.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 4.5 on page 113.

4.2. Segments, Rays, Angles, Triangles

Our first two new definitions are for segments and rays. Notice the use of coordinate functions in the definitions.

Definition 25 segment, ray

words and symbols: "segment A, B", denoted $\overline{AB}$, and "ray A, B", denoted $\overrightarrow{AB}$
usage: A and B are distinct points.
meaning: Let f be a coordinate function for line $\overleftrightarrow{AB}$ with the property that $f(A) = 0$ and $f(B)$ is positive. (The existence of such a coordinate function is guaranteed by Theorem 11 (Ruler Placement Theorem).)

- Segment $\overline{AB}$ is the set $\overline{AB} = \{P \in \overleftrightarrow{AB} \text{ such that } 0 \le f(P) \le f(B)\}$.
- Ray $\overrightarrow{AB}$ is the set $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} \text{ such that } 0 \le f(P)\}$.

additional terminology:

- Points A and B are called the *endpoints* of segment $\overline{AB}$.
- Point A is called the *endpoint* of ray $\overrightarrow{AB}$.
- The *length* of a segment is defined to be the distance between the endpoints. That is, $length(\overline{AB}) = d(A, B)$. As mentioned in Definition 20, many books use the symbol AB to denote $d(A, B)$. Thus we have the following choice of notations:
$$length(\overline{AB}) = d(A, B) = AB$$

In drawings, it is obvious that a segment $\overline{AB}$ is a subset of ray $\overrightarrow{AB}$. The same is true in our Axiomatic Geometry. We will need to use that fact frequently in this book, and so we should state it as a theorem. It is so easy to prove that it should be called a corollary of the definitions.

Theorem 18 (Corollary) Segment $\overline{AB}$ is a subset of ray $\overrightarrow{AB}$.
Proof
(1) Suppose that point P is an element of segment $\overline{AB}$. (We must show that P is also an element of ray $\overrightarrow{AB}$.)
(2) A and B are distinct points. **(Justify.)**
(3) There exists a coordinate function f for line $\overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B)$ is positive. **(Justify.)**
(4) The inequality $0 \le f(P) \le f(B)$ is true. **(Justify.)**
(5) So the inequality $0 \le f(P)$ is true. **(Justify.)**
(6) So point P is an element of ray $\overrightarrow{AB}$. **(Justify.)**
End of Proof

A simple but important fact about rays has to do with the notation.

Theorem 19 about the use of different second points in the symbol for a ray.
If $\overrightarrow{AB}$ and C is any point of $\overrightarrow{AB}$ that is not A, then $\overrightarrow{AB} = \overrightarrow{AC}$.

Proof (for readers interested in advanced topics and for graduate students)
The proof is left to the reader as an exercise.

The next three definitions make use of segments and rays.

Definition 26 Opposite rays are rays of the form $\overrightarrow{BA}$ and $\overrightarrow{BC}$ where $A * B * C$.

Definition 27 angle
words: "angle A, B, C"

symbol: $\angle ABC$
usage: A, B, C are non-collinear points.
meaning: Angle A, B, C is defined to be the following set: $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$
additional terminology: Point B is called the *vertex* of the angle. Rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are each called a *side* of the angle.

Definition 28 triangle
words: "triangle A, B, C"
symbol: ΔABC
usage: A, B, C are non-collinear points.
meaning: Triangle A, B, C is defined to be the following set: $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$
additional terminology: Points A, B, C are each called a *vertex* of the triangle. Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CA}$ are each called a *side* of the triangle.

Notice that the definitions of angle and triangle make no mention of any sort of "interior" of those objects. We will formulate a notion of interior for angles and triangles in the next chapter.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 4.5 on page 113.

4.3. Segment Congruence

In drawings, we have a notion of whether or not segments are the "same size". We determine whether or not two segments are the same size by measuring their lengths. In this chapter, we have introduced a notion of segment length into our abstract Neutral Geometry, so it will be easy to also introduce a notion of "same size" for segments. The term that we will use is *congruent*.

Definition 29 segment congruence

Two line segments are said to be congruent if they have the same length. The symbol $\cong$ is used to indicate this. For example $\overline{AB} \cong \overline{CD}$ means $length(\overline{AB}) = length(\overline{CD})$. Of course, this can also be denoted $d(A, B) = d(C, D)$ or $AB = CD$.

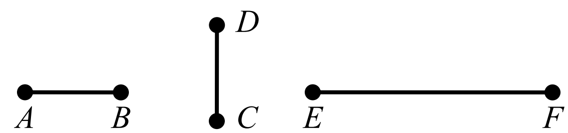
It is fruitful to discuss the properties of segment congruence using the terminology of relations. For that, we should first review some of that terminology.

First, we will review the basic notation for a relation.

Suppose that we are given a relation $\mathcal{R}$ on some set A. When x and y are elements of the set A (that is, when $x \in A$ and $y \in A$), the symbol ${}_x\mathcal{R}_y$ is read as a sentence *"x is related to y"*.

For example, let A be the set of integers, and define $\mathcal{R}$ to be the *less than* relation. That is, define the symbol ${}_x\mathcal{R}_y$ to mean $x < y$. Then the symbol ${}_5\mathcal{R}_7$ would mean "$5 < 7$", which is a true sentence. On the other hand, the symbol ${}_7\mathcal{R}_5$ would mean "$7 < 5$", which is a false sentence. In general, the sentence ${}_x\mathcal{R}_y$ may be true or false. Writing down the sentence, either in words or in symbols, does not imply that the sentence is true.

Notice that segment congruence is a relation on the set of line segments. That is, the symbol $\cong$ gets put in between two other symbols that represent line segments, and the resulting sentence may be true or it may be false. For in the example, for the drawing below, the sentence $\overline{AB} \cong \overline{CD}$ is true, but the sentence $\overline{AB} \cong \overline{EF}$ is false.



Now that we have reviewed the basic notation for relations, we can discuss three important properties that general relations may or may not have. The are the reflexive, symmetric, and transitive properties.

Definition 30 reflexive property
words: Relation $\mathcal{R}$ is *reflexive*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: Element of set A is related to itself.
abbreviated version: For every $x \in A$, the sentence ${}_x\mathcal{R}_x$ is true.
More concise abbreviaton: $\forall x \in A,\ {}_x\mathcal{R}_x$

Definition 31 symmetric property
words: Relation $\mathcal{R}$ is *symmetric*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: If x is related to y, then y is related to x.
abbreviated version: For every $x, y \in A$, if ${}_x\mathcal{R}_y$ is true then ${}_y\mathcal{R}_x$ is also true.
More concise abbreviaton: $\forall x, y \in A, if\ {}_x\mathcal{R}_y\ then\ {}_y\mathcal{R}_x$

Definition 32 transitive property
words: Relation $\mathcal{R}$ is *transitive*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: If x is related to y and y is related to z, then x is related to z.
abbreviated: For every $x, y, z \in A$, if ${}_x\mathcal{R}_y$ is true and ${}_y\mathcal{R}_z$ is true, then ${}_x\mathcal{R}_z$ is also true.
More concise abbreviaton: $\forall x, y \in A, if\ {}_x\mathcal{R}_y\ and\ {}_y\mathcal{R}_z\ then\ {}_x\mathcal{R}_z$

Consider some familiar mathematical relations in light of these definitions.

For example, the "less than" relation, denoted by the symbol "<" is not reflexive and not symmetric, but it is transitive.

On the other hand, the "less than or equal to" relation, denoted by the symbol "$\leq$" is reflexive, is not symmetric, and is transitive.

There is a special name for a relation that has all three of the above properties.

Definition 33 equivalence relation
words: Relation $\mathcal{R}$ is an *equivalence relation*.

usage: $\mathcal{R}$ is a relation on some set A.
meaning: $\mathcal{R}$ is reflexive and symmetric and transitive.

Now that we have reviewed the terminology of relations, we are ready to understanding the wording of the following theorem.

Theorem 20 Segment congruence is an equivalence relation.

It is worthwhile to discuss the proof of Theorem 20 in some detail, because there are a few similar theorems later in the book. You will be asked to prove those theorems on your own.

Proof of Theorem 20

Part 1: Prove that segment congruence is reflexive.

Segment congruence is a relation on the set of all line segments. To prove that the relation has the reflexive property, we must prove that the relation of segment congruence satisfies the definition of the reflexive property.

The reflexive property is the following sentence:

For every $x \in A$, the sentence ${}_x\mathcal{R}_x$ is true.

But in our case, the relation $\mathcal{R}$ is the relation of segment congruence, and the set A is the set of all line segments. That is, we would rewrite the reflexive property as follows:

For every line segment $\overline{CD}$, the sentence $\overline{CD} \cong \overline{CD}$ is true.

Observe that this sentence is a universal statement about line segments. Recall what we have discussed about proof structure for universal statements: we must start the proof by introducing a given generic line segment, and we must end the proof by saying that the segment is congruent to itself.

Proof Structure for Part 1:
(1) Suppose a line segment $\overline{CD}$ is given.

(some steps here)

(*) Conclude that the sentence $\overline{CD} \cong \overline{CD}$ is true. (We will need to justify this.)
End of proof for Part 1

Now our job is to figure out how to fill in the missing details of this proof. In general, many mathematical proofs involve some sort of "leap", some sort of inspiration. That can be very intimidating. You might look at the proof structure above and have no idea how to proceed forward from statement (1). But that is not surprising, because statement (1) does not really give you any clue as to how to proceed. The key is to look ahead to the final step, and try to work backwards from there.

The final statement says that $\overline{CD} \cong \overline{CD}$. We need to justify that statement. Keep in mind that the statement of segment congruence is a defined statement (See Definition 29 of segment congruence on page 104.) The statement $\overline{CD} \cong \overline{CD}$ really just means that $d(C,D) = d(C,D)$. So if we want to prove that $\overline{CD} \cong \overline{CD}$, we have only one way to do it: we must first prove that $d(C,D) = d(C,D)$ and then use Definition 29 to say that $\overline{CD} \cong \overline{CD}$. Thus we have narrowed the gap in our proof structure. Our proof now looks like this:

Proof Structure for Part 1:

(1) Suppose a line segment $\overline{CD}$ is given.

(some steps here)

(*) The sentence $d(C,D) = d(C,D)$ is true. (We will need to justify this.)
(*) Conclude that the sentence $\overline{CD} \cong \overline{CD}$ is true. (By the previous statement and Definition 29 of line segment congruence.)

End of proof for Part 1

Now keep in mind that the distance between points C and D is a *real number*. We know that every real number is equal to itself, by the *reflexive property of real number equality*. That gives us a way to justify the sentence $d(C,D) = d(C,D)$. But we also need to have first introduced the real number that we are talking about. That is, we need to state how we know that the real number $d(C,D)$ even exists. For this, we can use an axiom.

It turns out that we are now ready to completely fill the gap in our proof structure. Our final proof looks like this:

Proof for Part 1:

(1) Suppose a line segment $\overline{CD}$ is given.
(2) The real number $d(C,D)$ exists. (by axiom <N4>)
(3) $d(C,D) = d(C,D)$. (By (2) and the reflexive property of real number equality.)
(4) Conclude that $\overline{CD} \cong \overline{CD}$. (By the (3) and Definition 29 of line segment congruence.)

End of proof for Part 1

A quick summary of our approach in writing the Proof for Part 1 will help us in the rest of the proof.

- We began by translating the statement of the reflexive property (from the definition) into a statement about segment congruence.
- We observed that the statement to be proven was a universal statement about line segments. That gave us the idea of a "frame" for the proof, the first and last statements of the proof.
- We noted that the final statement of the proof needed to be justified. Since the final statement used a defined expression—segment congruence—we knew that the only way to justify it would be to use the definition of segment congruence. That gave us the justification for the final statement and also indicated what would need to be the next-to-final statement.

- Our goal was to prove the reflexive property of segment congruence. A key statement turned out to be statement (3), which was justified by the reflexive property of real number equality. This makes sense, because segment congruence was defined in terms of equality of lengths, which are real numbers.

We will find that a similar approach will work in the Proof for Part 2. In particular, we will find that a key step the proof of the symmetry property of segment congruence will use the symmetry property of real number equality.

Part 2: Prove that segment congruence is symmetric.

We will follow the approach that we used in the Proof of Part 1.

Translate the statement of the symmetric property into a statement about segment congruence.

The symmetric property is the following sentence:

For every $x, y \in A$, if ${}_x\mathcal{R}_y$ is true then ${}_y\mathcal{R}_x$ is also true.

But in our case, the relation $\mathcal{R}$ is the relation of segment congruence, and the set A is the set of all line segments. That is, we would rewrite the symmetric property as follows:

For every pair of line segments $\overline{CD}, \overline{EF}$, if $\overline{CD} \cong \overline{EF}$ then $\overline{EF} \cong \overline{CD}$.

Determine the "frame" for the proof.

Observe that this sentence is a universal statement about line segments. Recall what we have discussed about proof structure for universal statements: we must start the proof by introducing a pair of given generic line segments $\overline{CD}, \overline{EF}$ that are known to have the property $\overline{CD} \cong \overline{EF}$, and we must end the proof by saying that $\overline{EF} \cong \overline{CD}$.

Proof Structure for Part 2:

(1) Suppose line segments $\overline{CD}, \overline{EF}$ are given and that $\overline{CD} \cong \overline{EF}$.

(some steps here)

(*) Conclude that the sentence $\overline{EF} \cong \overline{CD}$ is true. (We will need to justify this.)

End of proof for Part 2

Work backward from the final statement.

The final statement says that $\overline{EF} \cong \overline{CD}$. We need to justify that statement. As we did in the Proof of Part 1, we note that the statement of segment congruence is a defined statement (See Definition 29 of segment congruence on page 104.) The statement $\overline{EF} \cong \overline{CD}$ really just means that $d(E, F) = d(C, D)$. So if we want to prove that $\overline{EF} \cong \overline{CD}$, we have only one way to do it:

We must first prove that $d(E,F) = d(C,D)$ and then use Definition 29 to say that $\overline{CD} \cong \overline{CD}$. Thus we have narrowed the gap in our proof structure. Our proof now looks like this:

Proof Structure for Part 2:
(1) Suppose line segments $\overline{CD}, \overline{EF}$ are given and that $\overline{CD} \cong \overline{EF}$.

(some steps here)

(*) The sentence $d(E,F) = d(C,D)$ is true. (We will need to justify this.)
(*) Conclude that the sentence $\overline{EF} \cong \overline{CD}$ is true. (By the previous statement and Definition 29 of line segment congruence.)
End of Proof for Part 2

Something New in the Proof for Part 2: Work forward from the first statement.

So far in our building of the Proof for Part 2, we have followed the same steps that we followed when building our Proof for Part 1, and we have succeeded in narrowing the gap in the proof. But at this point, the Proof for Part 2 becomes more complicated than the Proof for Part 1.

Look back two paragraphs at our discussion of the final statement of the proof. We observed that the final statement of the proof was $\overline{EF} \cong \overline{CD}$, and that this statement used a defined term (segment congruence). The only way to justify that statement was by using the definition line segment congruence (Definition 29). That gave us an idea for how to work backwards from the final statement of the proof.

Now consider the first statement of the proof. It says that $\overline{CD} \cong \overline{EF}$. This statement uses a defined term (segment congruence). The only thing that we can do with the information that the segments are congruent is to use the definition of segment congruence to translate the information into a different form. This gives us an idea of how to work forward from statement (1). Our proof now looks like this:

Proof Structure for Part 2:
(1) Suppose line segments $\overline{CD}, \overline{EF}$ are given and that $\overline{CD} \cong \overline{EF}$.
(2) The sentence $d(C,D) = d(E,F)$ is true. (By statement (1) and Definition 29 of line segment congruence.)

(some steps here)

(*) The sentence $d(E,F) = d(C,D)$ is true. (We will need to justify this.)
(*) Conclude that the sentence $\overline{EF} \cong \overline{CD}$ is true. (By the previous statement and Definition 29 of line segment congruence.)
End of proof for Part 2

Finishing touches

So far, we have narrowed the gap in our proof substantially simply by observing things that we have no choice about. That is, we had no choice about the "frame" of the proof: the frame was dictated by the fact that the statement to be proven was a universal statement. And we had no choice about how to work backwards from the final statement: since the final statement was a statement involving a defined term (segment congruence), we had no choice but to work backwards using the definition of that term. Similarly, we had no choice about how to work forwards from the first statement.

But how do we close the gap? How do we make the jump from the statement $d(C,D) = d(E,F)$ to the statement $d(E,F) = d(C,D)$? That's easy: the two symbols $d(C,D)$ and $d(E,F)$ represent real numbers. If we wanted to, we could rename them x and y. If we know that $x = y$, then we also automatically know that $y = x$ by the *symmetric property of real number equality*. So we are ready to close the gap and finish our Proof of Part 2. The final proof looks like this:

Proof Structure for Part 2:

(1) Suppose line segments $\overline{CD}, \overline{EF}$ are given and that $\overline{CD} \cong \overline{EF}$.
(2) $d(C,D) = d(E,F)$ (By (1) and Definition 29 of line segment congruence.)
(3) $d(E,F) = d(C,D)$ is true. (By (2) and the symmetric property of real number equality.)
(4) Conclude that the sentence $\overline{EF} \cong \overline{CD}$ is true. (By (3)and Definition 29 of line segment congruence.)

End of proof for Part 2

Notice that goal was to prove the symmetric property of segment congruence. A key statement turned out to be statement (3), which was justified by the symmetric property of real number equality. This makes sense, because segment congruence was defined in terms of equality of lengths, which are real numbers. And it is exactly the same thing that happened in our proof of the reflexive property of segment congruence.

In the exercises, you will be asked to prove the transitive property of segment congruence. You will find the the same approach that we used in the proofs of the reflexive and symmetric properties will work for the proof of the transitive property.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 4.5 on page 113.

4.4. Segment Midpoints

In drawings, the midpoint of a line segment is a point that is on the segment and that is equidistant from the endpoints of the segment. We are comfortable with the idea that in drawings, every segment has exactly one midpoint. We would like to be able to define the midpoint of an abstract line segment in an analogous way, and to know that each abstract line segment has exactly one midpoint. It is easy enough to make a definition of a midpoint of an abstract line segment. But we will need a couple of theorems to state that every line segment has exactly one midpoint. We start with the definition:

Definition 34 midpoint of a segment

Words: M is a midpoint of Segment A, B.

Meaning: M lies on $\overleftrightarrow{AB}$ and $MA = MB$.

Now we need a theorem stating that every line segment has exactly one midpoint. Surprisingly, that will take a bit of work. It is helpful to start by thinking about how we actually find the midpoint of a segment in a drawing: We use a ruler to find the point whose ruler marking is midway between the ruler markings of the endpoints of the segment. Our abstract Neutral Geometry allows us to do the same sort of thing with a coordinate function on a line. This next theorem makes the process precise and proves that it does indeed find a midpoint. The proof is included for readers interested in advanced topics, and for graduate students.

Theorem 21 About a point whose coordinate is the average of the coordinates of the endpoints. Given $\overline{AB}$, and Point C on line $\overleftrightarrow{AB}$, and any coordinate function f for line $\overleftrightarrow{AB}$, the following are equivalent:

(i) The coordinate of point C is the average of the coordinates of points A and B. That is, $f(C) = \frac{f(A)+f(B)}{2}$.

(ii) Point C is a midpoint of segment $\overline{AB}$. That is, $CA = CB$.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that $\overline{AB}$ and Point C on line $\overleftrightarrow{AB}$, and a coordinate function f for line $\overleftrightarrow{AB}$ are given.

Part I:Show that (i) ➔ (ii).

(2) Suppose that Statement (i) is true. That is, suppose that $f(C) = \frac{f(A)+f(B)}{2}$.

(3) Observe that

$$d(C,A) = |f(C) - f(A)| = \left|\frac{f(A)+f(B)}{2} - f(A)\right| = \left|\frac{f(B)-f(A)}{2}\right|$$

and that

$$d(C,B) = |f(C) - f(B)| = \left|\frac{f(A)+f(B)}{2} - f(B)\right| = \left|\frac{f(A)-f(B)}{2}\right| = \left|\frac{f(B)-f(A)}{2}\right|.$$

That is, $CA = CB$. So statement (ii) is true.

End of Part I

Part II:Show that (ii) ➔ (i).

(4) Suppose that Statement (ii) is true. That is, suppose that $CA = CB$.

(5) It must be true that $A * C * B$. (If $C * A * B$, then Theorem 16 (Betweenness of points on a line is related to the distances between them.) would tell us that

$$d(C,B) = d(C,A) + d(A,B)$$

But this combined with the fact that $d(C,A) = d(C,B)$ would imply that $d(A,B) = 0$. This is impossible. Similarly, we can show that $A * B * C$ is also impossible.)

(6) Therefore, we can use Theorem 16 to say that

$$\begin{aligned} d(A,B) &= d(A,C) + d(C,B) \\ &= d(A,C) + d(A,C) \quad \big(because\ d(C,A) = d(C,B)\big) \\ &= 2d(A,C) \end{aligned}$$

and hence $d(A,C) = \frac{d(A,B)}{2}$.

(7) We also know that $f(A) * f(C) * f(B)$. **(Justify.)**

(8) Either $f(A) < f(C) < f(B)$ or $f(B) < f(C) < f(A)$. **(Justify.)**

Case I: $\boldsymbol{f(A) < f(C) < f(B)}$

(9) Suppose that $f(A) < f(C) < f(B)$.

(10) Then

$$\begin{aligned} f(C) &= f(C) - f(A) + f(A) \quad (trick) \\ &= |f(C) - f(A)| + f(A) \quad (because\ f(C) - f(A)\ is\ positive) \\ &= d(A,C) + f(A) \quad (justify) \\ &= \frac{d(A,B)}{2} + f(A) \quad (justify) \\ &= \frac{|f(B) - f(A)|}{2} + f(A) \quad (justify) \\ &= \frac{f(B) - f(A)}{2} + f(A) \quad (because\ \ f(B) - f(A)\ is\ positive) \\ &= \frac{f(B) + f(A)}{2} \end{aligned}$$

so statement (i) is true.

Case II: $f(B) < f(C) < f(A)$

(11) Suppose that $f(B) < f(C) < f(A)$. Observe that these inequalities look just like the inequalities in step (9), but with the A and B symbols interchanged. We could build a string of equations here by interchanging the A and B symbols in step (10) and the result would be the equation $f(C) = \frac{f(B)+f(A)}{2}$. So statement (i) is true.

Conclusion of Cases

(12) We see that statement (i) is true in either case.

End of Part II and End of Proof

The previous theorem has an immediate corollary that tells us that if a point has a coordinate that is the average of the coordinates of the endpoints when using a particular coordinate function, then the same thing will be true when using any other coordinate function.

Theorem 22 Corollary of Theorem 21.

Given $\overline{AB}$, and Point C on line $\overleftrightarrow{AB}$, and any coordinate functions f and g for line $\overleftrightarrow{AB}$, the following are equivalent:

(i) $f(C) = \frac{f(A)+f(B)}{2}$.

(ii) $g(C) = \frac{g(A)+g(B)}{2}$.

Proof (for readers interested in advanced topics and for graduate students)

The proof is left to the reader.

We are now able to state a theorem about the existence and uniqueness of midpoints.

Theorem 23 Every segment has exactly one midpoint.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose segment $\overline{AB}$ is given.

Existence of a midpoint

(2) There exists a coordinate function f for line $\overleftrightarrow{AB}$. **(Justify.)**

(3) There exists a point M with coordinate $f(M) = \frac{f(A)+f(B)}{2}$. **(Justify.)**

(4) M is a midpoint of segment $\overline{AB}$. **(Justify.)**

Uniqueness of the midpoint.

(5) Suppose that point M and N are midpoints of segment $\overline{AB}$.

(*) some missing steps

(#) Therefore, points M and N must be the same point. **(Justify.)**
End of proof

We have just seen that the idea of the existence of a unique midpoint—a very simple concept in a drawing—was a little tricky to nail down in our axiomatic geometry. The following theorem expresses another fact that is fairly simple to understand, but whose proof is fairly detailed. The proof is included for readers interested in advanced topics, and for graduate students.

Theorem 24 Congruent Segment Construction Theorem.
Given a segment $\overline{AB}$ and a ray $\overrightarrow{CD}$, there exists exactly one point E on ray $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Proof (for readers interested in advanced topics and for graduate students)
(1) There exists a coordinate function for line $\overleftrightarrow{CD}$ such that $f(C) = 0$ and $f(D)$ is positive. **(Justify.)**
Show that such a point *E* exists.
(2) Let $E = f^{-1}\big(d(A,B)\big)$. That is, E is the point on line $\overleftrightarrow{CD}$ whose coordinate $f(E)$ is the real number $d(A,B)$.
(3) Then $d(C,E) = |f(E) - f(C)| = |d(A,B) - 0| = d(A,B)$. **(Justify.)** This tells us that $\overline{CE} \cong \overline{AB}$.
Show that the point *E* is unique.
(4) Suppose that E' is a point on ray $\overrightarrow{CD}$ such that $\overline{CE'} \cong \overline{AB}$. (we will show that E' must be E.)

(*) some missing steps

(#) Point E' must be E. **(Justify.)**
End of Proof

The final two theorems of the section are simple to understand and to prove. Their proofs are left to you as exercises.

Theorem 25 Congruent Segment Addition Theorem.
If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.

Theorem 26 Congruent Segment Subtraction Theorem.
If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$ then $\overline{BC} \cong \overline{B'C'}$.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 4.5 on page 113.

4.5. Exercises for Chapter 4

Exercises for Section 4.1 Betweenness

[1] Suppose $A * B * C$. Need A, B, C be distinct points? Explain.

[2] (Advanced) Justify the steps in the proof of Theorem 14 (Lemma about betweenness of coordinates of three points on a line) which was presented on page 99.

[3] (Advanced) Justify the steps in the proof of Theorem 15 (Properties of Betweenness for Points) Statements (A), (B), (C) which was presented on page 100. Write your own proof of Statement (D).

[4] (Advanced) Justify the steps in the proof of Theorem 16 (Betweenness of points on a line is related to the distances between them.) Part I which was presented on page 101. Write your own proof Part II.

Exercises for Section 4.2 Segments, Rays, Angles, Triangles

[5] Justify the steps in the proof of Theorem 18 ((Corollary) Segment $\overline{AB}$ is a subset of ray $\overrightarrow{AB}$.), which was presented on page 103.

[6] (Advanced) Prove Theorem 19 (about the use of different second points in the symbol for a ray.), which was presented on page 103.

[7] Refer to the definition of angle. (Definition 27 on page 103.) In the three figures below, the object in figure (1) consists of just the points inside, the object in figure (2) consists of just the rays, and the object in figure (3) consists of the rays and the points inside. Which figure is a valid illustration of an angle $\angle ABC$? Explain.

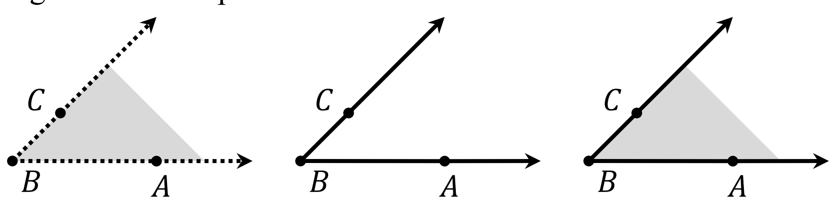


figure (1) figure (2) figure (3)

[8] Jack says that $\angle ABC$ is the negative of $\angle CBA$. Zack insists that $\angle ABC$ and $\angle CBA$ are the same thing. Who is right, Jack or Zack? Explain.

[9] Does either of the figures below depict a valid angle $\angle ABC$? Explain why or why not.

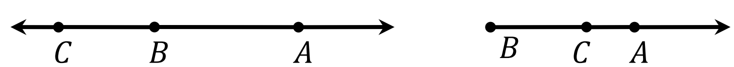


figure (1) figure (2)

[10] Refer to the definition of triangle. (Definition 28 on page 104.) In the three figures below, the object in figure (1) consists of just the points inside, the object in figure (2) consists of just the segments, and the object in figure (3) consists of the segments and the points inside. Which figure is a valid illustration of an triangle ΔABC? Explain.

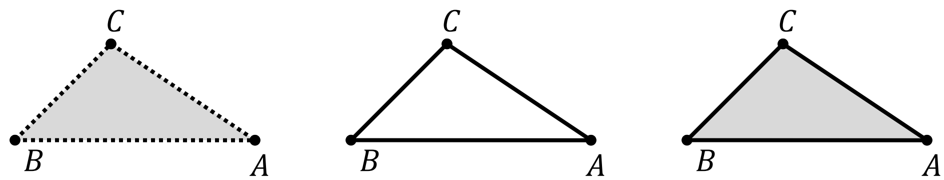


figure (1) figure (2) figure (3)

[11] Jack says that the symbols ΔABC and ΔBAC mean different things. Zack insists that ΔABC and ΔBAC mean the same thing. Who is right, Jack or Zack? Explain.

Exercises for Section 4.3 Segment Congruence

[12] In the reading, you explored part of the proof of Theorem 20 (Segment congruence is an equivalence relation.), presented on page 106. You saw detailed discussions of Proof Part 1 (segment congruence is reflexive) and Proof Part 2 (segment congruence is symmetric). Write your own proof of Part 3: segment congruence is transitive.

[13] Recall that in our axiomatic geometry, parallel lines are defined to be lines that do not intersect. (Definition 11, found on page 35) Using the terminology of relations, we could think of "parallel" as a relation on the set of all lines. Is the parallel relation an equivalence relation? Explain.

[14] How are parallel lines usually defined in analytic geometry? Is the usual parallel relation from analytic geometry an equivalence relation? Explain.

Exercise for Section 4.4 Segment Midpoints

[15] (Advanced) Justify the steps in the proof of Theorem 21 (About a point whose coordinate is the average of the coordinates of the endpoints.), which was presented on page 111.

[16] (Advanced) Prove Theorem 22 (Corollary of Theorem 21.), which was presented on p. 112.

[17] (Advanced) Justify the steps in the proof of Theorem 23 (Every segment has exactly one midpoint.), which was presented on page 112. Write the missing steps.

[18] (Advanced) Justify the steps in the proof of Theorem 24 (Congruent Segment Construction Theorem.), which was presented on page 113. Write the missing steps.

[19] Prove Theorem 25 (Congruent Segment Addition Theorem.), which was presented on p. 113.

[20] Prove Theorem 26 (Congruent Segment Subtraction Theorem.), presented on page 113.

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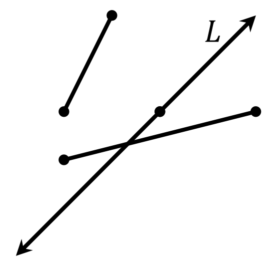
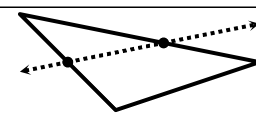
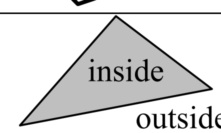
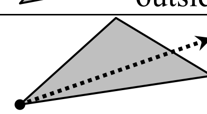
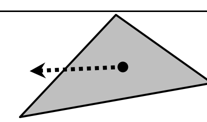
5.Neutral Geometry III: The Separation Axiom

5.1. Introduction to The Separation Axiom and Half-Planes

In the previous chapter, we saw that the Neutral Geometry Axioms of Incidence and Distance ensured that there is a notion of distance in our abstract geometry that agrees with our notions about distance in drawings. In the current chapter, we will discuss other kinds of behavior of drawings and show how the Neutral Geometry axioms ensure that our abstract geometry will exhibit the same behaviors. Here are five examples of familiar behavior of drawings:

Example #1: Consider the way a drawn line L "splits" the plane of a drawing. Notice three things: (1) Any point must be either on line L or on one side of it or the other. (2) If two points are on the same side of line L, then the segment connecting those two points also lies on the same side of L and does not intersect L. (3) If two points are on opposite sides of line L, then the segment connecting those two points will intersect line L.	
Example #2: In a drawing, any line that intersects a side of a triangle at a point that is not a vertex must also intersect at least one of the opposite sides.	
Example #3: Drawn triangles have an "inside" and an "outside".	
Example #4: In a drawing, any ray drawn from a vertex into the inside of a triangle must hit the opposite side of the triangle somewhere and go out.	
Example #5: In a drawing, a triangle cannot enclose a ray. The endpoint and part of the ray may fit inside the triangle, but the ray must poke out somewhere.	

It turns out that in the list of axioms for Neutral Geometry, a single new axiom will ensure that our axiomatic geometry will have these same behaviors and many others. That axiom is called the *Plane Separation Axiom*.

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following properties:

(i) The three sets L, H_1, H_2 form a partition of the set of all points.
(ii) Each of the half-planes is convex.
(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

This axiom describes the behavior that we observed in the first example, above. But the wording of the axiom doesn't quite match the wording of the observation. To understand the wording of the axiom, we will need to review a bit of set terminology.
For example, what about behavior (1) in Example #1 above?

(1) Any point must be either on line L or on one side of it or the other.

There is no statement in Axiom <N6> that is worded this way. But consider property (i) in Axiom <N6>:

(i) The three sets L, H_1, H_2 form a partition of the set of all points.

Remember that the set of all points is denoted by the symbol $\mathcal{P}$. To say that *the three sets* L, H_1, H_2 *form a partition of the set of all points* means that the union $L \cup H_1 \cup H_2 = \mathcal{P}$ and also that the three sets L, H_1, H_2 are mutually disjoint. In other words, every point P lies in exactly one of the sets L, H_1, H_2. So we see that property (i) in Axiom <N6> guarantees that our axiomatic geometry will have behavior analogous to behavior (1) in Example #1 above.

Now consider behavior (2) in Example #1 above:

(2) If two points are on the same side of line L, then the segment connecting those two points also lies on the same side of L and does not intersect L.

There is no statement in Axiom <N6> that is worded this way. But consider property (ii) in Axiom <N6>:

(ii) Each of the half-planes is convex.

The word *convex* appears in Axiom <N6>, but not in our observation about drawings. The definition is probably familiar to you:

Definition 35 convex set

- Without names: A set is said to be *convex* if for any two distinct points that are elements of the set, the segment that has those two points as endpoints is a subset of the set.
- With names: Set S is said to be convex if for any two distinct points $P, Q \in S$, the segment $\overline{PQ} \subset S$.

The fact that each of the half-planes is convex tells us that if two points P, Q lie in one of the half-planes, then segment $\overline{PQ}$ will be contained in that half plane. The fact that the sets L, H_1, H_2 are mutually disjoint then tells us that segment $\overline{PQ}$ will not intersect line L. So we see that properties (1) and (ii) in Axiom <N6> guarantee that our axiomatic geometry will have behavior analogous to behavior (2) in Example #1 above.

Finally, consider behavior (3) in Example #1 above:

(3) If two points are on opposite sides of line L, then the segment connecting those two points will intersect line L.

Property (iii) in Axiom <N6> seems to be about the same sort of behavior. It says,

(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

But notice that the wording of the observation about the drawn example is a bit different from the wording of the axiom. The observation about drawings mentions points being on the "same side" or on "opposite sides", while the Axiom mentions the idea of a point being in one half-plane or the other. This difference can be bridged by defining the terms of *same-side* and *opposite side*. I will also introduce the term edge of a half-plane.

Definition 36 same side, opposite side, edge of a half-plane.

Two points are said to lie on the *same side* of a given line if they are both elements of the same half-plane created by that line. The two points are said to lie on *opposite sides* of the line if one point is an element of one half-plane and the other point is an element of the other. The line itself is called the *edge of the half-plane*.

With the definition of the terminology of *same side* and *opposite side*, we see that property (iii) in Axiom <N6> guarantees that our axiomatic geometry will have behavior analogous to behavior (3) in Example #1 above.

Throughout the rest of the course, we will frequently use axiom <N6> in proofs to justify statements that say that two particular points are on the same side of some line, or on different sides of some line, or that some line segment connecting two points does not intersect some line, or that it does intersect some line. In those situations, it will be useful to observe that the statements <N6>(ii) and <N6>(iii) can be written as conditional statements and that those conditional statements have contrapositive statements that are logically equivalent to the original statements.

Here is Axiom <N6>(ii) restated in conditional form, along with the contrapositive.

<N6> (ii) If points P and Q are in the same half plane,
then segment $\overline{PQ}$ does not intersect line L.

<N6> (ii) (contrapositive) If segment $\overline{PQ}$ does intersect line L (at a point between P and Q),
then points P and Q are in different half planes.

Here is Axiom <N6>(iii) restated in conditional form, along with the contrapositive.

<N6>(iii) If points P and Q are in different half planes,
then segment $\overline{PQ}$ intersects line L (at a point between P and Q).

<N6>(iii) (contrapositive) If segment $\overline{PQ}$ does not intersect line L,
then points P and Q are in the same half plane.

Digression about using Conditional Statements and their Contrapositives

A digression about using conditional statements and their contrapositives. Suppose that two axioms are stated in the form of conditional statements, as follows.

- Axiom <100>: If the dog is blue, then the car is red.
- Axiom <101>: If the car is red, then the bear is hungry.

The contrapositives of these two axioms would be the following statements:

- Axiom <100> (contrapositive): If the car is not red, then the dog is not blue.
- Axiom <101> (contrapositive): If the bear is not hungry, then the car is not red.

Remember that the contrapositive statements are logically equivalent to the original statements.

Suppose that we wanted to prove that the car is red. Then clearly, we would use Axiom <100>. Our strategy would be to first prove somehow that the dog is blue, and then use Axiom <100> to say that the car is red. Note that we would not use Axiom <101> to prove that the car is red. Axiom <101> tells us something about the situation where we already know that the car is red. (It tells us that in this situation, the bear is hungry.)

Now, suppose that we wanted to prove that the car was not red. It is important to realize that Axiom <100> does not help us in this case! If we want to prove that the car is not red, then we need to use Axiom <101> (contrapositive). Our strategy would be to first prove somehow that the bear is not hungry, and then use Axiom <101> (contrapositive) to say that the car is not red.

This discussion is relevant to your use of Neutral Geometry Axiom <N6> (ii) and <N6> (iii) in proofs. For instance, suppose that you want to prove that two points P and Q are in the same half plane of some line L. You should not use Axiom <N6> (ii). That axiom says something about the situation where you already know that points P and Q are in the same half plane. (It says that in that situation, segment $\overline{PQ}$ does not intersect line L. Rather, you should use Axiom <N6> (iii) (contrapositive). Your strategy should be to prove somehow that segment $\overline{PQ}$ does not intersect line L, and then use Axiom <N6> (iii) (contrapositive) to say that points P and Q are in the same half plane.

End of Digression about using Conditional Statements and their Contrapositives

In the remaining sections of this chapter, we will see how the Separation Axiom <N6> ensures that our abstract geometry will have behavior like that observed in Examples #2 through #5 above. Before going on, though, it is worthwhile to study one simple-sounding theorem that is just about half-planes. You will justify the steps and make drawings in a class drill.

Theorem 27 Given any line, each of its half-planes contains at least three non-collinear points.

Proof

(1) Given any line, call it L_1. **(Make a drawing.)**

Introduce points A and B.

(2) There exist two distinct points on L_1. **(Justify.)** Call them A and B. **(Make a new drawing.)**

Part I: Introduce Half-Plane H_C and show that it contains three non-collinear points.

(3) There exists a point not on L_1. **(Justify.)** Call it C. **(Make a new drawing.)**
(4) Point C lies in one of the two half-planes determined by line L_1. **(Justify.)** Call it H_C. **(Make a new drawing.)**

Introduce line L_2.
(5) There exists a unique line passing through A and C. **(Justify.)**
(6) The line passing through A and C is not L_1. **(Justify.)** So it must be new. Call it L_2. **(Make a new drawing.)**

Introduce line L_3.
(7) There exists a unique line passing through B and C. **(Justify.)**
(8) The line passing through B and C is not L_1 or L_2. **(Justify.)** So it must be new. Call it L_3. **(Make a new drawing.)**

Introduce point D.
(9) There exists a point such that $A * C * Point$. **(Justify.)**
(10) This point cannot be the same as any of our previous three points. **(Justify.)** So it must be a new point. Call it D. So $A * C * D$. **(Make a new drawing.)**
(11) Point D is in half-plane H_C. **(Justify.)**

Introduce point E.
(12) There exists a point such that $B * C * Point$. **(Justify.)**
(13) This must be a new point. **(Justify.)** Call it E. So $B * C * E$. **(Make a new drawing.)**
(14) Point E is in half-plane H_C. **(Justify.)**

Conclusion of Part I:
(15) Points C and D and E are non-collinear. **(Justify.)**

Part II: Introduce Half-Plane H_F and show that it contains three non-collinear points.

Introduce point F.
(16) There exists a point such that $C * A * Point$. **(Justify.)**
(17) This must be a new point. **(Justify.)** Call it F. So $C * A * F$. **(Make a new drawing.)**
(18) Point F is not in half-plane H_C. **(Justify.)**. Let H_F be the half-plane containing F.

Introduce point G.
(19) There exists a point such that $A * F * Point$. **(Justify.)**
(20) This must be a new point. **(Justify.)** Call it G. So $A * F * G$. **(Make a new drawing.)**
(21) Point G is in half-plane H_F. **(Justify.)**

Introduce point H.
(22) There exists a point such that $C * B * Point$. **(Justify.)**
(23) This must be a new point. **(Justify.)** Call it H. So $C * B * H$. **(Make a new drawing.)**
(24) Point H is in half-plane H_F. **(Justify.)**

Conclusion of Part II:
(25) Points F and G and H are non-collinear. **(Justify.)**

End of Proof

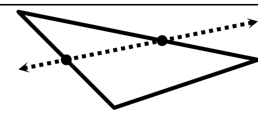
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.2. Theorems about lines intersecting triangles

The only geometric objects that we know about so far are points, lines, rays, segments, angles, and triangles. To more fully understand the significance of Separation Axiom <N6>, we will in the next four sections of this book explore what the Separation Axiom tells us about the relationships between those objects. In this section we will study two theorems about lines intersecting triangles.

Recall Example #2 of the previous section:

Example #2: In a drawing, any line that intersects a side of a triangle at a point that is not a vertex must also intersect at least one of the opposite sides.



The first of the two theorems that we will study in this section is known as Pasch's Theorem. It shows that abstract triangles and lines in our axiomatic geometry will have the same kind of behavior that Example #2 observes in drawings. You will justify the steps in a class drill.

Theorem 28 (Pasch's Theorem) about a line intersecting a side of a triangle between vertices
If a line intersects the side of a triangle at a point between vertices, then the line also intersects the triangle at another point that lies on at least one of the other two sides.

Proof

(1) Suppose that line L intersects side $\overline{AB}$ of ΔABC at a point D such that $A * D * B$.
(2) Points A and B are on opposite sides of line L. **(Justify.)** Let H_A and H_B be their respective half-planes.
(3) Exactly one of the following statements is true. **(Justify.)**
(i) C lies on L. **(Make a drawing for case (i).)**
(ii) C is in H_A. **(Make a drawing for case (ii).)**
(iii) C is in H_B. **(Make a drawing for case (iii).)**

Case (i)

(4) If C lies on L, then L intersects both $\overline{AC}$ and $\overline{BC}$ at point C. **(Justify.)**

Case (ii)

(5) If C is in H_A, then points B and C lie on opposite sides of L. **(Justify.)**
(6) In this case, L will intersect $\overline{BC}$ at a point between B and C. **(Justify.)**

Case (iii)

(7) If C is in H_B, then points A and C lie on opposite sides of L. **(Justify.)**
(8) In this case, L will intersect $\overline{AC}$ at a point between A and C. **(Justify.)**

Conclusion of cases

(9) In every case, we see that L intersects $\overline{AC}$ or $\overline{BC}$ or both.

End of Proof

The second of the two theorems that we will study in this section is about about a line intersecting two sides of a triangle. You will prove it in an exercise.

Theorem 29 about a line intersecting two sides of a triangle between vertices
If a line intersects two sides of a triangle at points that are not vertices, then the line cannot intersect the third side.

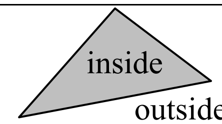
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.3. Interiors of angles and triangles

In the previous section, we studied two theorems that showed how the Separation Axiom <N6> could dictate some of the behavior of lines intersecting triangles.

Recall our discussion of familiar behavior of drawings in Section 5.1. Our third example was:

Example #3: Drawn triangles have an "inside" and an "outside".



In the current section, we will introduce the concept of the interiors and exteriors of abstract angles and triangles in our axiomatic geometry.

Here are the definitions of angle interior and triangle interior.

Definition 37 Angle Interior

Words: The interior of $\angle ABC$.

Symbol: $Interior(\angle ABC)$

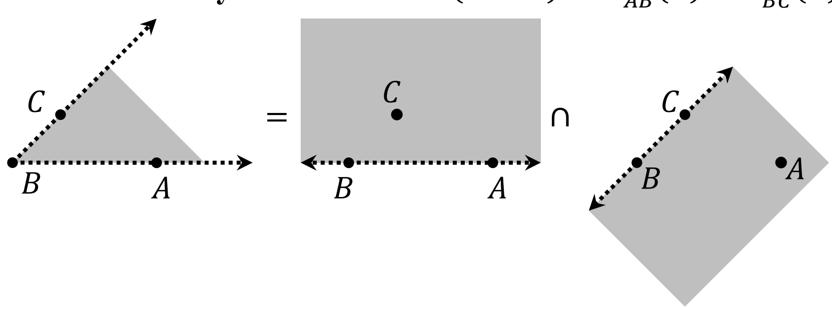
Meaning: The set of all points D that satisfy both of the following conditions.

Points D and A are on the same side of line $\overleftrightarrow{BC}$.

Points D and C are on the same side of line $\overleftrightarrow{AB}$.

Meaning abbreviated in symbols: $Interior(\angle ABC) = H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A)$

Drawing:



$Interior(\angle ABC) \ = \ H_{\overleftrightarrow{AB}}(C) \ \cap \ H_{\overleftrightarrow{BC}}(A)$

Related term: The *exterior* of $\angle ABC$ is defined to be the set of points that do not lie on the angle or in its interior.

Note that since it is the intersection of convex sets, an angle interior is a convex set.

Definition 38 Triangle Interior

Words: The interior of ΔABC.

Symbol: $Interior(\Delta ABC)$

Meaning: The set of all points D that satisfy all three of the following conditions.

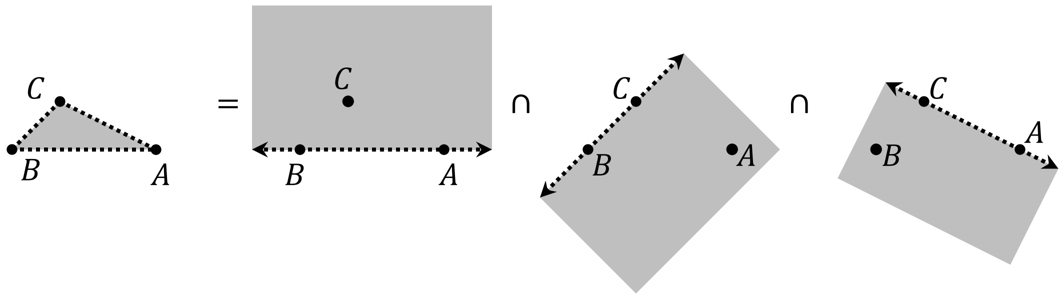
Points D and A are on the same side of line $\overleftrightarrow{BC}$.

Points D and B are on the same side of line $\overleftrightarrow{CA}$.

Points D and C are on the same side of line $\overleftrightarrow{AB}$.

Meaning abbreviated in symbols: $Interior(\Delta ABC) = H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A) \cap H_{\overleftrightarrow{CA}}(B)$.

Drawing:



$$Interior(\Delta ABC) \;=\; H_{\overleftrightarrow{AB}}(C) \;\cap\; H_{\overleftrightarrow{BC}}(A) \;\cap\; H_{\overleftrightarrow{CA}}(B)$$

Related term: The exterior of ΔABC is defined to be the set of points that do not lie on the triangle or in its interior.

Note that a triangle interior is a convex set. Also note that we can write

$$\begin{aligned} Interior(\Delta ABC) &= H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A) \cap H_{\overleftrightarrow{CA}}(B) \\ &= \left(H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A)\right) \cap \left(H_{\overleftrightarrow{BC}}(A) \cap H_{\overleftrightarrow{CA}}(B)\right) \cap \left(H_{\overleftrightarrow{CA}}(B) \cap H_{\overleftrightarrow{AB}}(C)\right) \\ &= Interior(\angle B) \cap Interior(\angle C) \cap Interior(\angle A) \end{aligned}$$

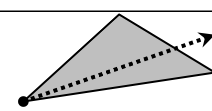
So the interior of a triangle is equal to the intersection of the interiors of its three angles.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.4. Theorems about rays and lines intersecting triangle interiors

Returning again to our discussion of familiar behavior of drawings in Section 5.1, recall that our fourth example was:

Example #4: In a drawing, any ray drawn from a vertex into the inside of a triangle must hit the opposite side of the triangle somewhere and go out.



In the current section, we will prove that in our axiomatic geometry, abstract rays and triangles behave the same way. The theorem, called the *Crossbar Theorem*, is a very difficult theorem to prove; we will need six preliminary theorems before we get to it! We start with a fairly simple theorem about rays that have an endpoint on a line but do not lie on the line. The proof is surprisingly tedious. Even so, I include the proof here and include the justification of its steps as an exercise, because studying its steps will provide a good review of a variety of concepts.

Theorem 30 about a ray with an endpoint on a line
If a ray has its endpoint on a line but does not lie in the line,
then all points of the ray except the endpoint are on the same side of the line.

Proof

(1) Let $\overrightarrow{AB}$ be a ray such that A lies on a line L and B does not lie on L, and let C be any point of $\overrightarrow{AB}$ that is not A. **(Make a drawing.)** (Our goal is to prove that point C lies on the same side of line L as point B.)

Introduce a special coordinate function and consider coordinates.

(2) Let M be line $\overleftrightarrow{AB}$.

(3) The only intersection of lines L and M is point A. **(Justify.)**

(4) There exists a coordinate function h for line M such that $h(A) = 0$ and $h(B)$ is positive. **(Justify.) (Make a new drawing.)**

(5) Using coordinate function h, the coordinate of point C is positive. That is $h(C)$ is positive. **(Justify.) (Make a new drawing.)**

Show that line segment $\overline{CB}$ does not intersect line L.

(6) Let D be any point on line segment $\overline{BC}$. **(Make a new drawing.)** Observe that point D lies on line M.

(7) Using coordinate function h, the coordinate of point D is positive. That is $h(D)$ is positive. **(Justify.) (Make a new drawing.)**

(8) Point D is not point A. **(Justify.)**

(9) Point D does not lie on line L. **(Justify.)** Therefore, line segment $\overline{BC}$ does not intersect line L.

Conclusion

(10) Conclude that points B and C are in the same half-plane of line L. **(Justify.)** That is, point C lies on the same side of line L as point B.

End of Proof

The preceeding theorem has two corollaries. (The word corollary has more than one usage in mathematics. I use the word here to mean a theorem whose proof is a simple application of some other theorem, with no other tricks.) You will be asked to prove both of them in exercises.

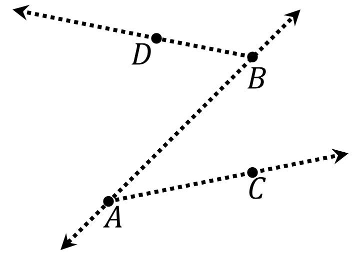
Theorem 31 (Corollary of Theorem 30) about a ray with its endpoint on an angle vertex
If a ray has its endpoint on an angle vertex and passes through a point in the angle interior, then every point of the ray except the endpoint lies in the angle interior.

Theorem 32 (Corollary of Theorem 30.) about a segment that has an endpoint on a line
If a segment that has an endpoint on a line but does not lie in the line,
then all points of the segment except that endpoint are on the same side of the line.

Here is a corollary of Theorem 32. You will be asked to prove it in an exercise.

Theorem 33 (Corollary of Theorem 32.) Points on a side of a triangle are in the interior of the opposite angle.
If a point lies on the side of a triangle and is not one of the endpoints of that side,
then the point is in the interior of the opposite angle.

The next theorem is not so interesting in its own right, but it gets used occasionally throughout the rest of the book in proofs of other theorems. Because of this, I call it a Lemma. Its name will help you remember what the Lemma says, by reminding you of the picture.



Theorem 34 The Z Lemma

If points C and D lie on opposite sides of line $\overleftrightarrow{AB}$, then ray $\overrightarrow{AC}$ does not intersect ray $\overrightarrow{BD}$.

Proof

(1) Suppose that points C and D lie on opposite sides of line $\overleftrightarrow{AB}$. Let H_C and H_D be their respective half-planes.

(2) Points A and B are distinct points on line $\overleftrightarrow{AB}$. (We cannot refer to line $\overleftrightarrow{AB}$ unless points A and B are distinct.)

(3) Every point of ray $\overrightarrow{AC}$ except endpoint A lies in half-plane H_C. **(Justify.)**

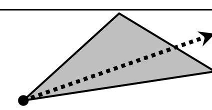
(4) Every point of ray $\overrightarrow{BD}$ except endpoint B lies in half-plane H_D. **(Justify.)**

(5) Ray $\overrightarrow{AC}$ does not intersect ray $\overrightarrow{BD}$. (by (2), (3), (4) and the fact that the three sets $\overleftrightarrow{AB}$ and H_C and H_D are mutually disjoint.)

End of Proof

Here, finally, is the Crossbar Theorem. Remember that it will prove that abstract triangles and rays have the sort of behavior that Example #4 described for drawings:

Example #4: In a drawing, any ray drawn from a vertex into the inside of a triangle must hit the opposite side of the triangle somewhere and go out.



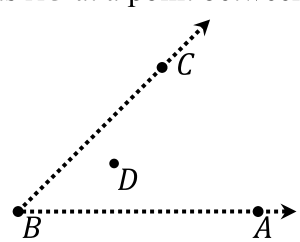
You will see that the proof relies on Pasch's Theorem (Theorem 28) and on repeated applications of the Z Lemma (Theorem 34).

Theorem 35 The Crossbar Theorem

If point D is in the interior of $\angle ABC$, then $\overrightarrow{BD}$ intersects $\overline{AC}$ at a point between A and C.

Proof

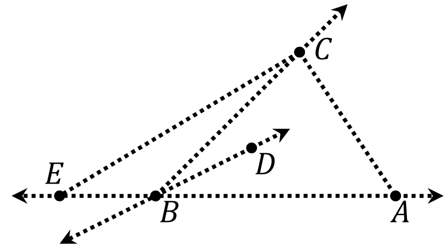
(1) Suppose that point D is in the interior of $\angle ABC$.



Introduce a point E that will allow us to use Pasch's Theorem.

(2) There exists a point E such that $A * B * E$ (by Theorem 15). Observe that line $\overleftrightarrow{BD}$ intersects side $\overline{EA}$ of ΔEAC at point B such that $E * B * A$.

(3) Line $\overleftrightarrow{BD}$ intersects ΔEAC at one other point. (by (2) and Pasch's Theorem (Theorem 28))

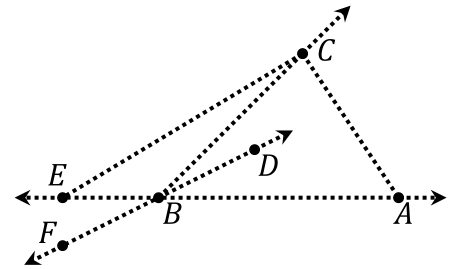


(4) The second point of intersection of line $\overleftrightarrow{BD}$ and triangle ΔEAC cannot be on line $\overline{AE}$ (because that would violate axiom <N2>), and it cannot be point C (because that would

also violate axiom <N2>) so line $\overleftrightarrow{BD}$ must intersect side either $\overline{AC}$ or side $\overline{EC}$, but not at one of the endpoints A, C, E.

Introduce a point *F*.

(5) There exists a point F such that $D * B * F$ (by Theorem 15). Observe that rays $\overrightarrow{BD}$ and $\overrightarrow{BF}$ are opposite rays.



Get more precise about particular rays intersecting particular segments

(6) Exactly one of the following must be true (by (4) and (5)).

(i) Ray $\overrightarrow{BF}$ intersects segment $\overline{AC}$ at a point between A and C.

(ii) Ray $\overrightarrow{BF}$ intersects segment $\overline{EC}$ at a point between E and C.

(iiii) Ray $\overrightarrow{BD}$ intersects segment $\overline{EC}$ at a point between E and C.

(iv) Ray $\overrightarrow{BD}$ intersects segment $\overline{AC}$ at a point between A and C.

Establish points that are on opposite sides of lines so that we can use the *Z* Lemma.

(7) Points C and D lie on the same side of line $\overleftrightarrow{AB}$ (by (1) and definition of angle interior).

(8) Points D and F lie on opposite sides of line $\overleftrightarrow{AB}$ (because line $\overleftrightarrow{AB}$ intersects segment $\overline{DF}$ at point B and point B is between points D and F, by (5)).

(9) Therefore, points C and F lie on opposite sides of line $\overleftrightarrow{AB}$ (by (7) and (8)).

(10) Points C and F also lie on opposite sides of line $\overleftrightarrow{BE}$ (since A, B, E are collinear by (2)).

(11) Points A and D lie on the same side of line $\overleftrightarrow{BC}$ (by (1) and definition of angle interior).

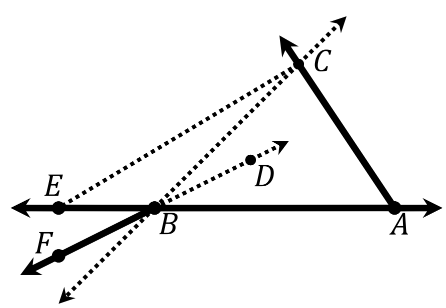
(12) Points A and E lie on opposite sides of line $\overleftrightarrow{BC}$ (because line $\overleftrightarrow{BC}$ intersects segment $\overline{AE}$ at point B and point B is between points A and E, by (2)).

(13) Therefore, points D and E lie on opposite sides of line $\overleftrightarrow{BC}$ (by (11) and (12)).

Use the *Z* Lemma three times.

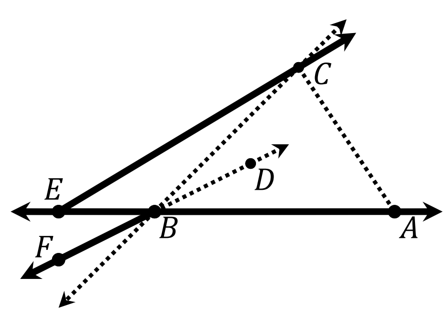
(14) Ray $\overrightarrow{BF}$ does not intersect ray $\overrightarrow{AC}$ (by the Z Lemma (Theorem 34) applied to line $\overleftrightarrow{AB}$ and points C and F that lie on opposite sides of it by (9))

(15) Therefore, ray $\overrightarrow{BF}$ does not intersect segment $\overline{AC}$ (because segment $\overline{AC}$ is a subset of ray $\overrightarrow{AC}$). So statement (6i) is not true.



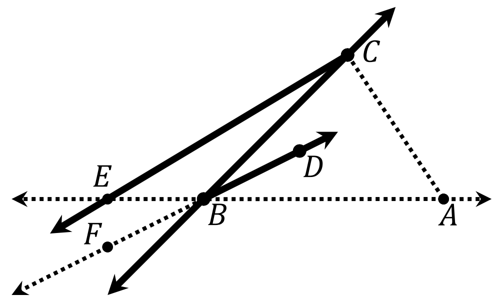
(16) Ray $\overrightarrow{BF}$ does not intersect ray $\overrightarrow{EC}$ (by the Z Lemma applied to line $\overleftrightarrow{BE}$ and points C and F that lie on opposite sides of it by (10))

(17) Therefore, ray $\overrightarrow{BF}$ does not intersect segment $\overline{EC}$. So statement (6ii) is not true.



(18) Ray $\overrightarrow{BD}$ does not intersect ray $\overrightarrow{CE}$ (by the Z Lemma applied to line $\overleftrightarrow{BC}$ and points D and E that lie on opposite sides of it by (13))

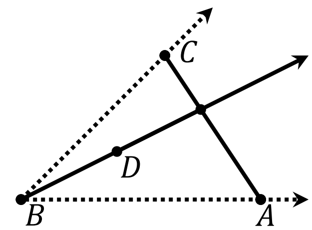
(19) Therefore, ray $\overrightarrow{BD}$ does not intersect segment $\overline{CE}$. So statement (6iii) is not true.



Conclusion

(20) Ray $\overrightarrow{BD}$ must intersect segment $\overline{AC}$ at a point between A and C (by (6), (15), (17), (19)).

End of Proof

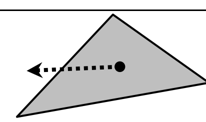


Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.5. A Triangle Can't Enclose a Ray or a Line

Returning for the last time to our discussion of familiar behavior of drawings in Section 5.1, recall that our fifth example was:

Example #5: In a drawing, a triangle cannot enclose a ray. The endpoint and part of the ray may fit inside the triangle, but the ray must poke out somewhere.



Similarly, in a drawing it is impossible to fit a line inside a triangle. The line must poke out at two points. In this section, we will prove that abstract rays, lines, and triangles have the same properties.

The first theorem of the section proves that a triangle cannot enclose a ray.

Theorem 36 about a ray with its endpoint in the interior of a triangle

If the endpoint of a ray lies in the interior of a triangle, then the ray intersects the triangle exactly once.

Proof (for readers interested in advanced topics and for graduate students)

The proof is left to the reader.

The second theorem of the section proves that a triangle cannot enclose a line.

Theorem 37 about a line passing through a point in the interior of a triangle

If a line passes through a point in the interior of a triangle, then the line intersects the triangle exactly twice.

Proof

(1) Suppose that line L passes through point P in the interior of ΔABC. **(Make a drawing.)**

(2) There exist points Q, R on line L such that $Q * P * R$. **(Justify. Update your drawing.)**
(3) Ray $\overrightarrow{PQ}$ intersects ΔABC exactly once. **(Justify.)**
(4) Ray $\overrightarrow{PR}$ intersects ΔABC exactly once. **(Justify.)**
(5) Line L intersects ΔABC exactly twice. **(Justify.)**
End of Proof

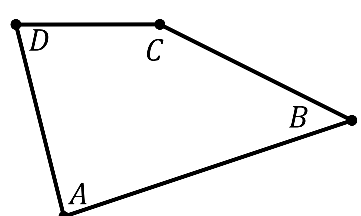
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.6. Convex quadrilaterals

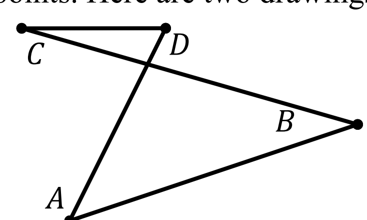
As mentioned at the start of Section 5.2 (Theorems about lines intersecting triangles) on page 121, the only geometric objects that we know about so far are points, lines, rays, segments, angles, and triangles. In the previous four sections, we explored what the Separation Axiom <N6> tells us about the relationships between those objects. In the current section we will introduce a new geometric object, the quadrilateral, and explore what Axiom <N6> tells us about the object.

Definition 39 quadrilateral
words: "quadrilateral A, B, C, D"
symbol: $\square ABCD$
usage: A, B, C, D are distinct points, no three of which are collinear, and such that the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ intersect only at their endpoints.
meaning: quadrilateral A, B, C, D is the set $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$
additional terminology: Points A, B, C, D are each called a *vertex* of the quadrilateral.
Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CD}$ and $\overline{DA}$ are each called a *side* of the quadrilateral.
Segments $\overline{AC}$ and $\overline{BD}$ are each called a *diagonal* of the quadrilateral.

Notice that it is not simply enough that we have four distinct points, no three of which are collinear. The requirement that the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ intersect only at their endpoints means that we must be careful how we name the points. Here are two drawings to illustrate.



quadrilateral

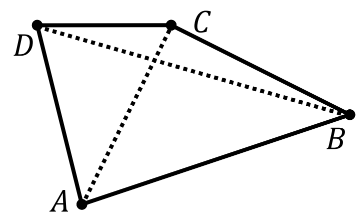
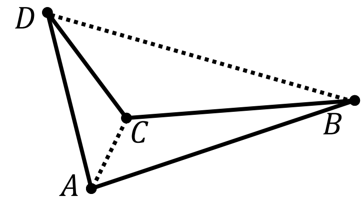


not a quadrilateral

The drawing on the right is not a quadrilateral because segments $\overline{AD}$ and $\overline{BC}$ intersect at a point that is not an endpoint of either segment. (Don't be fooled by the fact that the drawing does not have a large dot at that spot. Back in Chapters 1 and 2, when we were studying finite geometries, the only "points" in our drawings were the large dots that we intentionally drew. But now that we are studying Neutral Geometry, our lines contain an infinite number of points, one for each real number. So there is a point at the place where the drawn lines cross.)

In this section, we will classify quadrilaterals into one of two types. I will define the properties that distinguish the two types precisely later. Right now, I just want to draw examples of the two types and observe some things about the drawings. In particular, I want to observe whether

Statements (i), (ii), and (iii) are true or false. Shown below is a table that presents examples of the two types, along with observations about Statements (i), (ii), and (iii).

type of drawing:	type I	type II
drawing:		
Statement (i): All the points of any given side lie in the same half-plane of the line determined by the opposite side.	True	False, because there is a side whose points do not all lie in the same half-plane of the line determined by the opposite side.
Statement (ii): The diagonal segments intersect.	True	False, because the diagonal segments do not intersect (although the lines containing the diagonal segments do intersect.)
Statement (iii): Each vertex is in the interior of the opposite angle.	True	False, because there is a vertex that is not in the interior of the opposite angle.

Notice that in our two drawings, either all three statements (i), (ii), (iii) are true or they all are false. That is, the three statements are equivalent. The following theorem proves that in our axiomatic geometry, quadrilaterals behave the same way.

Theorem 38 Three equivalent statements about quadrilaterals

For any quadrilateral, the following statements are equivalent:

(i) All the points of any given side lie on the same side of the line determined by the opposite side.

(ii) The diagonal segments intersect.

(iii) Each vertex is in the interior of the opposite angle.

Proof that (i) ➔ (ii)

(1) Suppose that for quadrilateral $\square ABCD$, statement (i) is true.

Show that point $\boldsymbol{A}$ is in the interior of angle $\boldsymbol{\angle BCD}$ and use the Crossbar Theorem

(2) Points A and B are on the same side of line $\overleftrightarrow{CD}$. (by (1))

(3) Points A and D are on the same side of line $\overleftrightarrow{CB}$. (by (1))

(4) Point A is in the interior of angle $\angle BCD$. (by (2) and (3))

(5) Ray $\overrightarrow{CA}$ intersects segment $\overline{BD}$ at a point P between B and D. (by Theorem 35, The Crossbar Theorem)

Change letters to get more results of the same sort

(6) In steps (2) – (5), make the following replacements: $A \to B$ and $B \to C$ and $C \to D$ and $D \to A$ and $P \to Q$. The result will be a proof that point B is in the interior of angle $\angle CDA$ and that ray $\overrightarrow{DB}$ intersects segment $\overline{CA}$ at a point Q between C and A.

Wrap-up.

(7) Points P and Q must be the same point. (because lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ can only intersect at one point, by Theorem 1 (In Neutral Geometry, if L and M are distinct lines that intersect, then they intersect in only one point.))
(8) The point $P = Q$ lies on both segments $\overline{AC}$ and $\overline{BD}$. (by (5), (6),(7)) That is, the diagonals intersect. Conclude that statement (ii) is true.

End of proof that (i) ➔ (ii)

Proof that (ii) ➔ (iii)
(1) Suppose that for quadrilateral $\square ABCD$, statement (ii) is true. That is, the diagonal segments $\overline{AC}$ and $\overline{BD}$ intersect at some point P.

Show that point P lies in the interior of $\angle ABC$.
(2) Point P lies in the interior of $\angle ABC$. (by Theorem 33 ((Corollary of Theorem 32.) Points on a side of a triangle are in the interior of the opposite angle.))
(3) All the points of ray $\overrightarrow{BP}$ except B lie in the interior of $\angle ABC$. (by Theorem 31 ((Corollary of Theorem 30) about a ray with its endpoint on an angle vertex))
(4) Point D lies in the interior of $\angle ABC$. (because D is on ray $\overrightarrow{BP}$)

Change letters to get more results of the same sort
(7) In steps (2) – (4), make the following replacements: $A \to B$ and $B \to C$ and C→ D and $D \to A$. The result will be a proof that point A is in the interior of angle $\angle BCD$.
(8) Making analogous replacements, we can prove that point C is in the interior of angle $\angle DAB$ and that point D is in the interior of angle $\angle ABC$.
(9) Conclude that statement (iii) is true. (by (4),(7),(8))

End of proof that (ii) ➔ (iii)

Proof that (iii) ➔ (i)
(1) Suppose that for quadrilateral $\square ABCD$, statement (iii) is true. That is, each vertex lies in the interior of the opposite angle.

Consider point B.
(2) Point B lies in the interior of $\angle CDA$. (by (1))
(3) Points A and B are on the same side of line $\overleftrightarrow{CD}$. (by (2) and definition of angle interior)
(4) All the points of side $\overline{AB}$ are on the same side of line $\overleftrightarrow{CD}$. (by (3) and the fact that half-planes are convex.)
(5) Points B and C are on the same side of line $\overleftrightarrow{DA}$. (by (2) and definition of angle interior)
(6) All the points of side $\overline{BC}$ are on the same side of line $\overleftrightarrow{DA}$. (by (5) and the fact that half-planes are convex.)

Change letters to get more results of the same sort.
(7) In steps (2) – (6), make the following replacements: $A \to C$ and $B \to D$ and C→ A and $D \to B$. The result will be a proof that. All the points of side $\overline{CD}$ are on the same side of line $\overleftrightarrow{AB}$ and that all the points of side $\overline{DA}$ are on the same side of line $\overleftrightarrow{BC}$.

Conclusion.
(8) Conclude that all the points of any given side lie on the same side of the line determined by the opposite side. (by (4), (6), (7)) That is, Statement (i) is true.

End of proof that (iii) ➔ (i)

The preceeding theorem tells us that in our axiomatic geometry the three statements about quadrilaterals are indeed equivalent. Therefore, we can use any one of the three statements as the definition of a "Type I" quadrilateral. Most geometry books refer to this type of quadrilateral as a “convex quadrilateral,” so I will use that terminology. And I will use statement (i) as the definition.

Definition 40 convex quadrilateral

A *convex* quadrilateral is one in which all the points of any given side lie on the same side of the line determined by the opposite side. A quadrilateral that does not have this property is called *non-convex*.

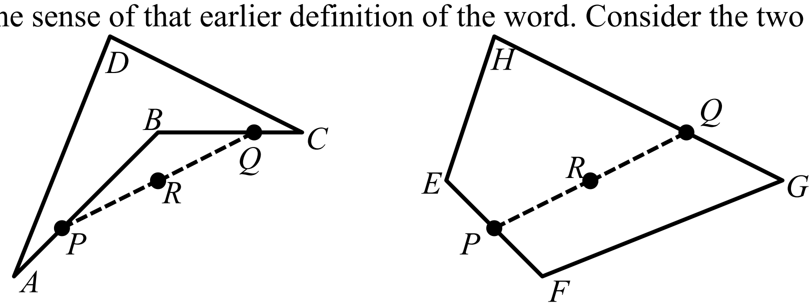
Digression to Discuss the Definition of a Convex Quadrilateral

It should be remarked the definition of a *convex quadrilateral* does not resemble our previous Definition of a convex set. Recall that definition.

Definition 35 of a *convex set*.

- Without names: A set is said to be *convex* if for any two distinct points that are elements of the set, the segment that has those two points as endpoints is a subset of the set.
- With names: Set S is said to be convex if for any two distinct points $P, Q \in S$, the segment $\overline{PQ} \subset S$.

This earlier definition of convex is familiar, resembling the kind of defnition of convex that you probably first encountered in grade school. It is natural to wonder why this early definition could not be used to define the notion of a convex quadrilateral. The reason is that a quadrilateral is never convex in the sense of that earlier definition of the word. Consider the two figures below.



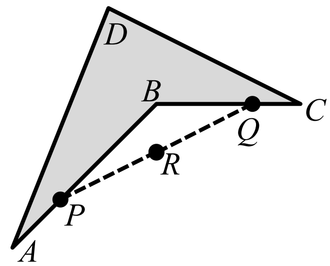
In both figures, notice that points *P,Q* lie on the quadrilateral, and point *R* does not lie on the quadrilateral. Using Definition 35 of a *convex set*, we would have to say that neither quadrilateral is convex. This is not very satisfying, because quad *EFGH* looks convex.

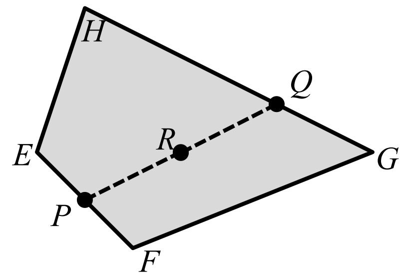
Clearly, the problem is that we are not considering the *interior* of the quadrilateral. Suppose that we used the following definition of convex quadrilateral:

One candidate for a definition of convex quadrilateral.

A quadrilateral would be called convex if the union of the quadrilateral and the interior of the quadrilateral are convex in the sense of Definition 35 of a *convex set*. That is, a quadrilateral *ABCD* would be said to be convex if for any two distinct points *P,Q* that lie on the quad, the segment $\overline{PQ}$ is contained in the union of the quad and its interior.

If we could use this definition, then quad *ABCD* would not be convex because, for instance, point *R* does not lie on the quad or in the interior of the quad. However quad *EFGH* would be considered convex because, for instance, point *R* *does* lie in the interior of the quad.





The candidate definition seems great. So why don't we use it? Well, it is surprisingly hard to give a make a definition of the *interior* of a quadrilateral using the terminology of our axiomatic geometry. Indeed, except for triangles, it is hard to make a definition of the interior of any polygon in our axiomatic geometry. (To see a bit of the difficulty, look ahead to Section 11.1, on page 243.) So we won't use a definition of convex quadrilateral (or convex polygon) that relies on the notion of the *interior*. That's why we use Definition 40 of Convex Quadrilateral.

End of Digression to Discuss the Definition of a Convex Quadrilateral

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 5.8 on page 134.

5.7. Plane Separation in High School Geometry Books

We were introduced to the SMSG axioms back in Section 3.10 Distance and Rulers in High School Geometry Books (on page 91) We observed that because they were written for a high school audience, the SMSG Postulates are written without some of the mathematical terminology that we are using in this book. It is interesting to compare the Plane Separation Postulate in theSMSG axiom system to our Neutral Geometry Axiom Plane Separation Axiom.

SMSG Postulate 9: (Plane Separation Postulate) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:

- each of the sets is convex
- if P is in one set and Q is in the other, then segment $\overline{PQ}$ intersects the line.

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following properties:
(i) The three sets L, H_1, H_2 form a partition of the set of all points.
(ii) Each of the half-planes is convex.
(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

Compare what the two axioms say about half-planes

- Our Neutral Geometry Axiom <N6> uses the terminology of *half-planes*, and has names for the half-planes. They are given the names H_1, H_2 and when they are discussed in statements (i) and (iii), they are referred to by name
- SMSG Postulate 9 does not use the term *half-plane*. The first sentence of the axiom simply says that there are "two sets". When the axiom refers to thes sets in the next two sentences, it can only refer to them as "one set" or "the other".

Compare what the two axioms say about partitions.

- Our Neutral Geometry Axiom <N6> says explicitly (in statement (i)) that the three sets L, H_1, H_2 form a partition of the set of all points.
- SMSG Postulate 9 does not seem to say anything about a partition. But actually, if you read the first sentence of SMSG Postulate 9 very carefully, you will realize that it does contain the essence of a partition. That is, if a point does not lie on the line, then it apparently lies in one of the "two sets". Well, actually, the SMSG postulate does not say that the point has to lie in *only one* of the two sets. That is, the SMSG postulate does not ever say that the two sets are disjoint. That is an error in the SMSG postulates.

5.8. Exercises for Chapter 5

Exercises for Section 5.1 Introduction to The Separation Axiom and Half-Planes

[1] The term *half-plane* is discussed in Section 5.1 which begins on page 117. Which of these figures is the best illustration of a half-plane in Neutral Geometry? Explain.

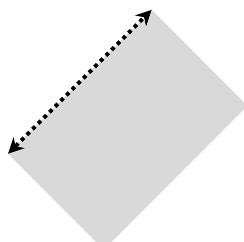


figure (1)

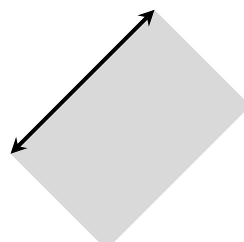


figure (2)

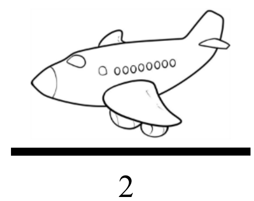
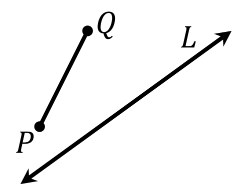


figure (3)

[2] In a drawing, if two points P, Q are on the same side of line L, then the segment connecting those two points also lies on the same side of L and does not intersect L. What guarantees that the same sort of thing will happen with abstract points and lines in Neutral Geometry? Explain.



[3] Suppose that in some proof, you want to prove that two points A and B are in the same half plane of some line L. What should be your strategy?

[4] Suppose that in some proof, you want to prove that two points A and B are *not* in the same half plane of some line L. What should be your strategy?

[5] Illustrate and justify the steps in the proof of Theorem 27 (Given any line, each of its half-planes contains at least three non-collinear points.) presented on page 120.

Exercises for Section 5.2 Theorems about lines intersecting triangles

[6] Illustrate and justify the steps in the proof of Theorem 28 ((Pasch's Theorem) about a line intersecting a side of a triangle between vertices) presented on page 122.

[7] Prove Theorem 29 (about a line intersecting two sides of a triangle between vertices) presented on page 122.

Exercises for Section 5.3 Interiors of angles and triangles

[8] Refer to the definition of Angle Interior (Definition 37, found on page 123). Suppose that in some proof, you want to prove that some point P is in the interior of some angle $\angle ABC$. What should be your strategy?

Exercises for Section 5.4 Theorems about rays and lines intersecting triangle interiors

[9] Illustrate and justify the steps in the proof of Theorem 30 (about a ray with an endpoint on a line) presented on page 124.

[10] Prove Theorem 31 ((Corollary of Theorem 30) about a ray with its endpoint on an angle vertex) presented on page 125.

[11] Prove Theorem 32 ((Corollary of Theorem 30.) about a segment that has an endpoint on a line) presented on page 125.

[12] Prove Theorem 33 ((Corollary of Theorem 32.) Points on a side of a triangle are in the interior of the opposite angle.) presented on page 125.

[13] Justify the steps in the proof of Theorem 34 (The Z Lemma) presented on page 126.

Exercises for Section 5.5 A Triangle Can't Enclose a Ray or a Line
[13] (Advanced) Prove Theorem 36 (about a ray with its endpoint in the interior of a triangle) presented on page 128.

[14] Illustrate and justify the steps in the proof of Theorem 37 (about a line passing through a point in the interior of a triangle) presented on page 128.

Exercises for Section 5.6 Convex quadrilaterals
[15] Illustrate the proof of Theorem 38 (Three equivalent statements about quadrilaterals) presented on page 130.

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6.Neutral Geometry IV: The Axioms of Angle Measurement

In Chapter 3, Neutral Geometry I: The Axioms of Incidence and Distance, we saw that our axiomatic geometry has a notion of distance that agrees with our notion of distance in drawings. It was the Axioms of Incidence and Distance (<N1> through <N5>) that specified the pertinent behavior.

In drawings, we measure the size of angles with a protractor. We would like to have a similar notion in our axiomatic geometry, with analogous behavior. In the current chapter, we will be introduced to the Axioms of Angle Measurement. These axioms will ensure that our axiomatic geometry will have a notion of angle measure that mimics our use of a protractor to measure angles in drawings. We will encounter no surprises there, mostly just introductions of terminology. Then, angle congruence will be defined in terms of equality of angle measure, just as segment congruence was defined in terms of equality of segment length in Chapter 3.

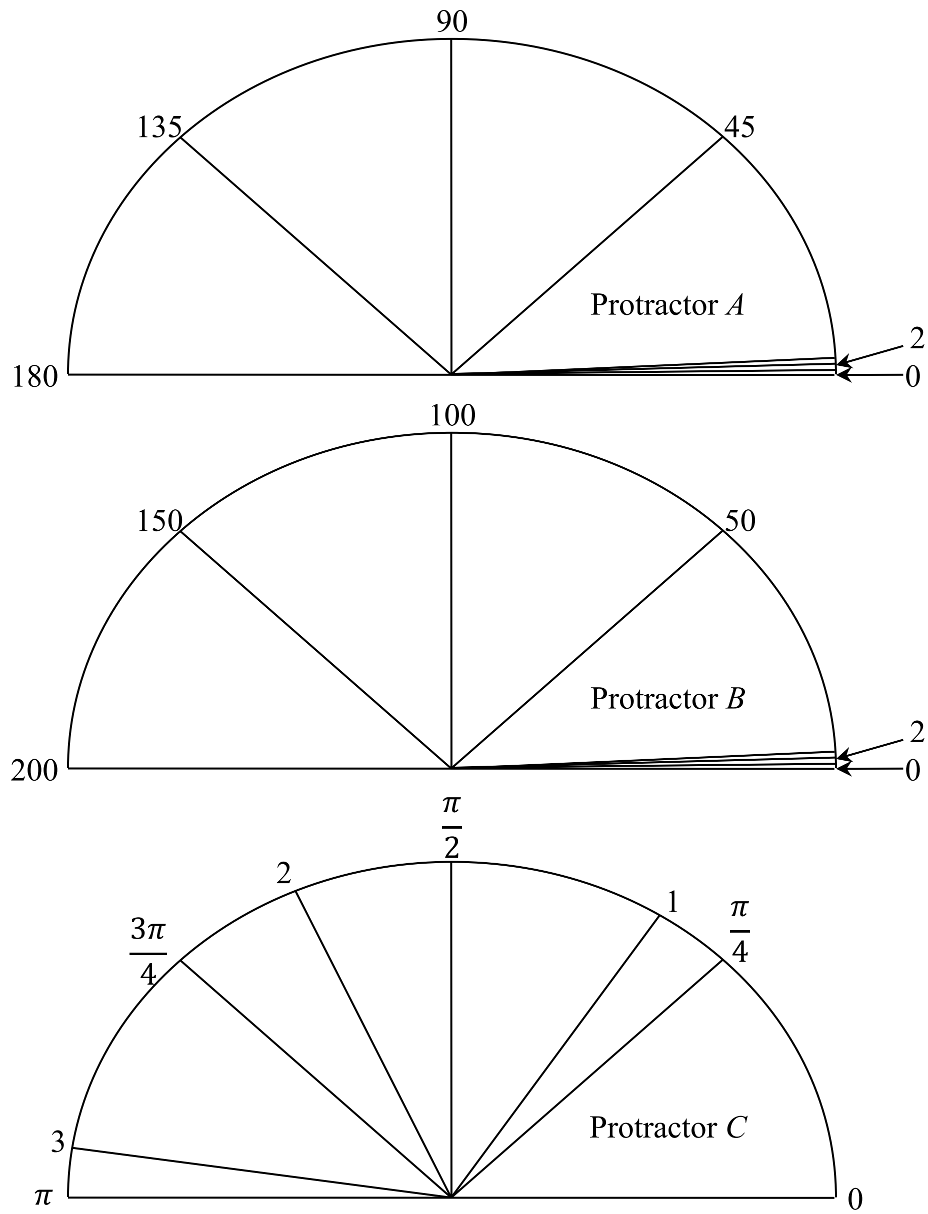
6.1. The Angle Measurement Axiom

When we measure the size of a drawn angle with a protractor, the result is a number between 0 and 180. We tack on the suffix "degrees", as in "37 degrees", or tack on a superscript open circle, such as in 37° as an abbreviation for "degrees". We want to have a notion of angle measure in our abstract geometry, but I would like it to not include the term "degree" or the superscript open circle,°. In order to do that, we need to consider what role the word "degrees" plays in our angle measurements in drawings.

Consider the protractors shown in the three drawings on the next page. Each drawing shows a few marks, but the actual protractors have more marks, as follows.

- Protractor *A* has equally spaced marks from 0 to 180.
- Protractor *B* has equally spaced marks from 0 to 200.
- Protractor *C* has equally spaced marks from 0 to π in increments of $\frac{\pi}{128}$. (These are all irrational numbers, and only a few of the marks are shown.) But this protractor also has a few extra marks thrown in, at the numbers 1, 2, and 3. (These are rational numbers.)

Suppose that somebody uses one of the protractors to measure an angle and then tells you that the angle has measure 2. Is that enough information to give you an idea of how the angle looks? Of course not. Without knowing which protractor was used to measure the angle, you can't visualize the angle. Only if you are told the number and also told which protractor was used, will the measurement be of any use. When the tag "degrees" or "gradians" or "radians" is put after a number, it is merely indicating that the number is a measure of an angle, and the tag is indicating which type of protractor is in use—type *A* or *B* or *C*.



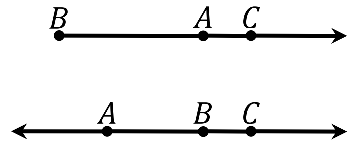
If it has been declared that all measurements of angle size in drawings are to be made with protractors only of type A, then there would be no need to add the tag "degrees" to the measurements. It would be known that any number that is a measure of an angle will be a number that was obtained using a protractor of type A, the one that goes from 0 to 180.

Now consider an equivalent situation in axiomatic geometry.

- To say that the measure of an angle is 2 $degrees$ or 2° would mean that the measure of the angle is 2 when using an angle measurement function with codomain $(0,180)$.
- To say that the measure of an angle is 2 $gradians$ would mean that the measure of the angle is 2 when using an angle measurement function with codomain $(0,200)$.
- To say that the measure of an angle is 2 $radians$ would mean that the measure of the angle is 2 when using an angle measurement function with codomain $(0,\pi)$.

In this book, we will only be using one angle measurement function. Denoted by the letter m, its codomain is the set $(0,180)$. Because we will only be using that one angle measurement function, we do not need to add a tag to our angle measurements. So instead of writing $m(\angle ABC) = 2°$, we will more simply write $m(\angle ABC) = 2$.

Note that the codomain of the angle measurement function m is $(0,180)$, not $[0,180]$. That is, the measure of an angle will always be a real number r such that $0 < r < 180$. What about $r = 0$ and $r = 180$? In drawings, if the measure of an angle $\angle ABC$ is 0, it means that the angle looks like the upper drawing at right. If the measure of an angle $\angle ABC$ is 180, it means that the angle looks like the lower drawing. In both cass, A, B, C are collinear. In our axiomatic geometry, the definition of *angle* includes the requirement that A, B, C be non-collinear. We will not have abstract angles that are analogous to the "zero angle" or "straight angle" from our drawings. So our angle measurement function m will never need to produce an output of 0 or 180.



Because m is a function, we should know how to describe it in function notation. For that, we will need a symbol for the set of all angles.

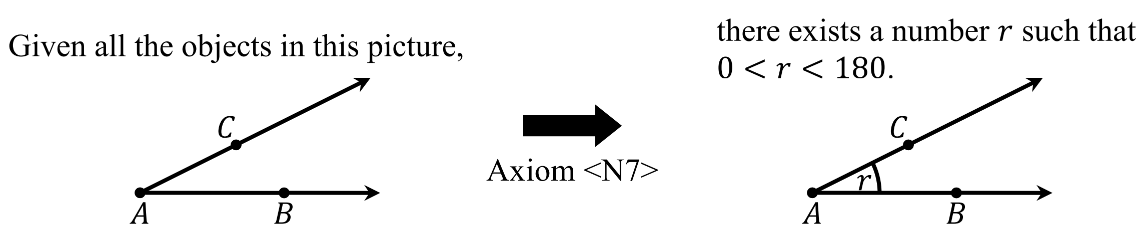
Definition 41 The set of all abstract angles is denoted by the symbol $\mathcal{A}$.

Using that symbol for the set of all angles, we can specify the function m as follows:

> <N7> (Angle Measurement Axiom) There exists a function $m: \mathcal{A} \to (0,180)$, called the Angle Measurement Function.

The notation indicates that m is a function that takes as input an angle and produces as output a real number between 0 and 180.

Even though the statement of the Angle Measurement Axiom is very simple, it is worthwhile to illustrate the statement with a drawing and to discuss the use of the axiom. Here is a drawing that illustrates the statement of the axiom:



There are a couple of very important things to observe about the statement of Axiom <N7>.

- Axiom <N7> does not give us the existence of the angle. The existence of the angle must have already been proved before Axiom <N7> can be used.
- Axiom <N7> does not tell us anything specific about the number r beyond the simple fact that the number exists and has a value in the range $0 < r < 180$. No other properties may be assumed for the number r. That means that whenever Axiom <N7> gets used in a proof, it marks the appearance of a *new* number r. The new number r has no relation to any number that has already appeared in the proof. Suppose, for instance, that a proof

step states that some angle has measure 37, or has measure equal to the measure of some other angle. The number 37 already existed in the proof, and the measure of the other angle already existed. Neither of those claims can be justified by Axiom <N7>. They would have to be justified in some other way.

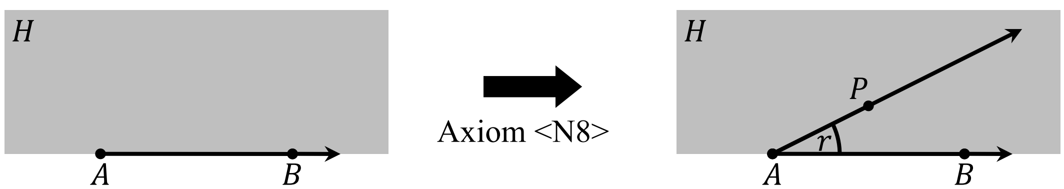
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 6.8 on page 153.

6.2. The Angle Construction Axiom

Here is the Angle Construction Axiom. Its statement is illustrated by the drawing below.

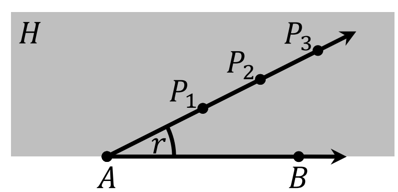
<N8> (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $m(\angle PAB) = r$.

Given all the objects in this picture — there exists exactly one ray like this:



and a number r such that $0 < r < 180$,

It is worth noting that the axiom says that there is exactly one *ray* $\overrightarrow{AP}$. It does not say that there is exactly one *point* P, and that is for a good reason. There are many such points P. Consider the drawing at right. Observe that



$$m(\angle P_1AB) = m(\angle P_2AB) = m(\angle P_3AB) = r$$

Notice that in the drawing, the three symbols $\overrightarrow{AP_1}, \overrightarrow{AP_2}, \overrightarrow{AP_3}$ all represent the same ray. Theorem 19 (about the use of different second points in the symbol for a ray.), found on page 103, tells us that the same thing is true in our abstract geometry. If abstract points P_2 and P_3 lie on abstract ray $\overrightarrow{AP_1}$, then the three symbols $\overrightarrow{AP_1}, \overrightarrow{AP_2}, \overrightarrow{AP_3}$ all represent the same abstract ray. The Angle Construction Axiom <N8> is worded to reflect this. The *abstract ray* $\overrightarrow{AP}$ is unique; the *abstract point* P is not.

There are a couple of very important things to observe about the statement of Axiom <N8>.

- Axiom <N8> does not give us the existence of the half-plane H, or the ray $\overrightarrow{AB}$ on the edge of the half-plane, or the number r. The existence of those things must have already been proved before Axiom <N8> can be used.
- Axiom <N8> does not tell us anything specific about the ray $\overrightarrow{AP}$ beyond the simple fact that the point P lies somewhere in half-plane H and that $m(\angle PAB) = r$. No other properties may be assumed for the point P. That means that whenever Axiom <N8> gets used in a proof, it marks the appearance of a *new* point P. The new point P has no

relation to any objects (except half-plane H and points A, B) that have already appeared in the proof.

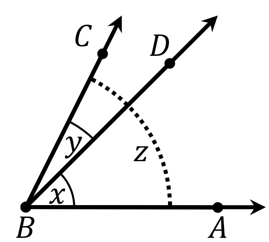
Another thing worth mentioning is that we can rephrase the Angle Construction Axiom using the terminology of the properties of functions. The axiom is telling us that under certain conditions, the angle measurement function is one-to-one and onto. To see how, recall that the symbol $\mathcal{A}$ represents the set of all angles. Suppose that A, B, C are non-collinear points. Then points A, B determine a unique line $\overleftrightarrow{AB}$, and point C is not on this line. The symbol H_C could be used to denote the half-plane created by line $\overleftrightarrow{AB}$ and containing point C. Ray $\overrightarrow{AB}$ is on the edge of the half-plane H_C. We could define the symbol $\mathcal{A}_{\overrightarrow{AB},H_C}$ to denote the set of all angles $\angle BAP$ such that $P \in H_C$. This is the set of all angles that have ray $\overrightarrow{AB}$ as one of their sides and have some ray $\overrightarrow{AP}$, where $P \in H_C$, as their other side. Of course, this collection of angles is a proper subset of the set of all angles. In symbols, we could write $\mathcal{A}_{\overrightarrow{AB},H_C} \subsetneq \mathcal{A}$. We can *restrict* the angle measurement function m to the smaller set $\mathcal{A}_{\overrightarrow{AB},H_C}$. That is, we can consider using the function m only on the angles in that set. The symbol $m\big|_{\mathcal{A}_{\overrightarrow{AB},H_C}}$ is used to denote the angle measurement function m restricted to the smaller set $\mathcal{A}_{\overrightarrow{AB},H_C}$. The angle construction axiom says that this *restricted* angle measurement function is both one-to-one and onto.

6.3. The Angle Measure Addition Axiom

In our drawings, we know that if a drawn point D is in the inside of a drawn angle $\angle ABC$, then

$$measure\ of\ \angle ABD + measure\ of\ \angle DBC = measure\ of\ \angle ABC$$

That is, if $measure\ of\ \angle ABD = x$ and $measure\ of\ \angle DBC = y$ and $measure\ of\ \angle ABC = z$, then $x + y = z$. An example is shown in the drawing at right. The angle addition axiom simply ensures that the same thing will happen with abstract angles in our axiomatic geometry.



<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle ABC$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$.

We should introduce some terminology for angles of the sort mentioned in Axiom <N9>.

Definition 42 adjacent angles

Two angles are said to be *adjacent* if they share a side but have disjoint interiors. That is, the two angles can be written in the form $\angle ABD$ and $\angle DBC$, where point C is not in the interior of $\angle ABD$ and point A is not in the interior of $\angle DBC$.

The following theorem states a small fact that is used frequently throughout the rest of the book. But simple as the fact is to state, it is surprisingly hard to prove. The proof is included here for readers interested in advanced topics and for graduate students.

Theorem 39 about points in the interior of angles

Given: points C and D on the same side of line $\overleftrightarrow{AB}$.
Claim: The following are equivalent:

(I) D is in the interior of $\angle ABC$.
(II) $m(\angle ABD) < m(\angle ABC)$.

Proof (for readers interested in advanced topics and for graduate students)
Proof that (I) ➔ (II)
(1) Suppose that D is in the interior of $\angle ABC$. **(Make a drawing.)**
(2) $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$. (by the Angle Measure Addition Axiom <N9>)
(3) $m(\angle ABD) < m(\angle ABC)$. (because $m(\angle DBC)$ is positive, by Axiom <N7>.)
End of proof that (1) ➔ (2)

Proof that (~I) ➔ (~II)
(4) Suppose that D is not in the interior of $\angle ABC$.
(5) There are two possibilities for point D. **(Justify.)**
(i) Point D lies on line $\overleftrightarrow{BC}$.
(ii) Points D and A lie on opposite sides of line $\overleftrightarrow{BC}$.

Case (i): Point D lies on line $\overleftrightarrow{BC}$.
(6) Suppose that point D lies on line $\overleftrightarrow{BC}$. **(Make a new drawing.)**
(7) Then ray $\overrightarrow{BD}$ is the same ray as $\overrightarrow{BC}$. **(Justify.)** So angle $\angle ABD$ is the same angle as $\angle ABC$. Thus, $m(\angle ABD) = m(\angle ABC)$. So Statement (II) is false in this case.

Case (ii): Points D and A lie on opposite sides of line $\overleftrightarrow{BC}$.
(8) Suppose that points D and A lie on opposite sides of line $\overleftrightarrow{BC}$. **(Make a new drawing.)**
(9) Segment $\overline{AD}$ intersects line $\overleftrightarrow{BC}$ at a point E between A and D. **(Justify.)**
(10) Points E and A are on the same side of line $\overleftrightarrow{BD}$. **(Justify.)**
(11) Points E and C are on the same side of line $\overleftrightarrow{BD}$. **(Justify.)**
(12) Points A and C are on the same side of line $\overleftrightarrow{BD}$. **(Justify.)**
(13) Point C is in the interior of $\angle ABD$. **(Justify.)**
(14) $m(\angle ABC) + m(\angle CBD) = m(\angle ABD)$. **(Justify.)**
(15) $m(\angle ABC) < m(\angle ABD)$. **(Justify.)** So Statement (II) is false in this case.

Conclusion of cases
(16) We see that Statement (II) is false in either case..
End of proof that (~I) ➔ (~II)

The idea of an angle bisector is very simple. Here's a definition.

Definition 43 angle bisector
An angle bisector is a ray that has its endpoint at the vertex of the angle and passes through a point in the interior of the angle, such that the two adjacent angles created have equal measure. That is, for an angle $\angle ABC$, a bisector is a ray $\overrightarrow{BD}$ such that $D \in interior(\angle ABC)$ and such that $m(\angle ABD) = m(\angle DBC)$.

The following theorem about the existence and uniqueness of angle bisectors is very important and its statement is not surprising. The proof relies on axioms <N8> and <N9> and on the previoius theorem about points in the interior of angles. You will justify it in a class drill. (Your justifications may refer to any prior theorem and use Neutral Axioms <N2> through <N9>. You may not use Axiom <N10>.)

Theorem 40 Every angle has a unique bisector.
Proof

(1) Suppose that angle $\angle ABC$ is given. **(Make a drawing.)**

Introduce special ray $\overrightarrow{BD}$ and show that it is a bisector of $\angle ABC$.

(2) The real number $m(\angle ABC)$ exists. **(Justify.)**

(3) Let $r = \frac{1}{2}m(\angle ABC)$. Let H_C be the half-plane created by line $\overleftrightarrow{AB}$ that contains point C. Observe that ray $\overrightarrow{BA}$ lies on the edge of this half-plane. **(Make a new drawing.)**

(4) There exists a ray $\overrightarrow{BD}$ such that $D \in H_C$ and $m(\angle ABD) = r$. **(Justify.) (Make a new drawing.)**

(5) Point D is in the interior of $\angle ABC$. (by statements (2), (3), and Theorem 39 II ➔ I) **(Make a new drawing.)**

(6) $m(\angle ABD) = m(\angle DBC)$. **(Justify. This will take 2 or 3 steps)**

(7) Ray $\overrightarrow{BD}$ is a bisector of $\angle ABC$. **(Justify.) (Make a new drawing.)**

Show that ray $\overrightarrow{BD}$ is the only bisector of $\angle ABC$.

(8) Suppose that ray $\overrightarrow{BE}$ is a bisector of $\angle ABC$. **(Make a new drawing.)**

(9) Point E is in the interior of $\angle ABC$ and $m(\angle ABE) = m(\angle EBC)$. **(Justify.) (Make a new drawing.)**

(10) $m(\angle ABE) = \frac{1}{2}m(\angle ABC)$. **(Justify.)**

(11) Points E and C are on the same side of line $\overleftrightarrow{AB}$. **(Justify.) (Make a new drawing.)**

(12) Point E is in half-plane H_C. **(Justify.) (Make a new drawing.)**

(13) Ray $\overrightarrow{BE}$ is the same ray as $\overrightarrow{BD}$. **(Justify.) (Make a new drawing.)**

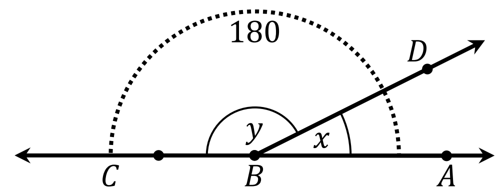
End of Proof

The above proof of Theorem 40 about the existence & uniqueness of an angle bisector depended on Theorem 39 about points in the interior of angles. Considered as a pair, those two theorems could be thought of as a long and difficult two-part proof of the existence & uniqueness of an angle bisector. Having gotten through that difficult two-part proof, I will now tell you the charming news that in the next chapter we will find that if we use the Axiom of Triangle Congruence, <N10>, there is a much easier proof of the existence and uniqueness of the angle bisector. Most books present only that easier proof. But it is worthwhile to study the two difficult proofs that we just studied, for at least three reasons: (1) they show that the existence & uniqueness of the angle bisector does not depend on the notion of triangle congruence, (2) the proof of Theorem 40 lets us see the Axiom of Separation used in conjunction with the Axioms of Angle Measurement and (3) seeing both the difficult proofs above and the forthcoming easier proof will provide us motivation to always search for alternate ways of proving things.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 6.8 on page 153.

6.4. The Linear Pair Theorem

Consider the angles $\angle ABD$ and $\angle DBC$ in the drawing at right below. Points A, B, C are collinear.



I have superimposed a protractor on the drawing. We see that if the protractor measure of drawn angle $\angle ABD$ is x
and the protractor measure of drawn angle $\angle DBC$ is y,
then $x + y = 180$.
We expect an analogous thing to happen in our abstract geometry. Will it?

To answer that question, it will help to introduce the terminology of a *linear pair* of angles.

Definition 44 linear pair
Two angles are said to be a *linear pair* if they share one side, and the sides that they do not share are opposite rays. That is, if the two angles can be written in the form $\angle ABD$ and $\angle DBC$, where $A * B * C$.

Now, what about the sum of the measures of the abstract angles in a linear pair? Notice that the drawn linear pair in our drawing could be considered as just a special case of the drawing of adjacent angles at the beginning of Section 6.3, on page 141. All that is special about the drawn linear pair is the additional fact that A, B, C are collinear. One might wonder if the Angle Addition Axiom <N9> will guarantee that abstract angles in an abstract linear pair will satisfy the equation

$$m(\angle ABD) + m(\angle DBC) = 180.$$

The answer to this question is no, for a fairly simple reason. The Angle Addition Axiom <N9> refers to three *angles*: $\angle ABD, \angle DBC, \angle ABC$. If two angles $\angle ABD$ and $\angle DBC$ form a linear pair, then the three points A, B, C are collinear, so they will not form an angle $\angle ABC$. So the Angle Addition Axiom does not apply in the situation where two angles $\angle ABD$ and $\angle DBC$ form a linear pair. So if we want to know that the equation $m(\angle ABD) + m(\angle DBC) = 180$ is true for angles in a linear pair, it will have to be specified in an another axiom or proven in a theorem.

It turns out that we don't need to specify in an additional axiom that the equation is true for angles in a linear pair, because we can prove it in a theorem.

Theorem 41 Linear Pair Theorem.
If two angles form a linear pair, then the sum of their measures is 180.

Proof (for readers interested in advanced topics and for graduate students)
(1) Suppose that two angles form a linear pair.
(2) The angles can be labeled $\angle ABD$ and $\angle DBC$ where $A * B * C$. **(Justify.) (Make a drawing.)**
(3) Let $x = m(\angle ABD)$ and $y = m(\angle DBC)$. **(Update your drawing.)**

Show that $x + y$ cannot be less than 180.
(4) Suppose that $x + y < 180$. (assumption)
(5) Let $r = x + y$.
(6) Then $0 < x < r < 180$. **(Justify.)**

(7) Let H_D be the half-plane created by line $\overleftrightarrow{AC}$ that contains point D. **(Make a new drawing.)**
(8) There exists a ray $\overrightarrow{BE}$ such that $E \in H_D$ and $m(\angle ABE) = r$. **(Justify.) (Make a new drawing.)**
(9) Observe that $\overrightarrow{BC}$ and $\overrightarrow{BE}$ are not the same ray (because E does not lie on line $\overleftrightarrow{AB}$).
(10) Point D is in the interior of $\angle ABE$. **(Justify.)**
(11) $m(\angle ABD) + m(\angle DBE) = r$. **(Justify.)**
(12) Therefore, $m(\angle DBE) = y$. **(Make a new drawing.)**
(13) Ray $\overrightarrow{BD}$ intersects segment $\overline{AE}$ at a point between A and E. **(Justify.)** We can label the point of intersection F. **(Make a new drawing.)**
(14) Points A and E are on opposite sides of line $\overleftrightarrow{BD}$. **(Justify.)**
(15) Points A and C are on opposite sides of line $\overleftrightarrow{BD}$. **(Justify)**
(16) Points E and C are on the same side of line $\overleftrightarrow{BD}$. **(Justify.)**
(17) Let H_C be the half-plane created by line $\overleftrightarrow{BD}$ that contains point C. Observe that this half-plane also contains point E. **(Make a new drawing.)**
(18) Rays $\overrightarrow{BC}$ and $\overrightarrow{BE}$ have $C \in H_C$ and $E \in H_C$ and $m(\angle DBC) = y$ and $m(\angle DBE) = y$ but they are not the same ray. (by steps (17), (3), (12), (9)) .**(Make a new drawing.)**
(19) Statement (18) is a contradiction. **(What does it contradict?)** Therefore, our assumption in step (4) was incorrect. That is, $x + y$ cannot be less than 180.

Show that $x + y$ cannot be greater than 180.
(20) Suppose that $x + y > 180$. (assumption) **(Make a new drawing.)**
(21) Then $0 < 180 - x < y < 180$. **(Justify.)**
(22) Let $r = 180 - x$. Then $0 < r < y < 180$.
(23) Let H_C be the half-plane created by line $\overleftrightarrow{BD}$ that contains point C. **(Make a new drawing.)**
(24) There exists a ray $\overrightarrow{BG}$ such that $G \in H_C$ and $m(\angle DBG) = r$. **(Justify.) (Make a new drawing.)**
(25) Point G is in the interior of $\angle DBC$. **(Justify.) (Make a new drawing.)**
(26) Points G and D are on the same side of line $\overleftrightarrow{BC}$ (which is also line $\overleftrightarrow{BA}$). **(Justify.) (Make a new drawing.)**
(27) Ray $\overrightarrow{BG}$ intersects segment $\overline{CD}$ at a point between C and D. **(Justify.)** We can label the point of intersection K. **(Make a new drawing.)**
(28) Points C and D are on opposite sides of line $\overleftrightarrow{BG}$. **(Justify.)**
(29) Points C and A are on opposite sides of line $\overleftrightarrow{BG}$. **(Justify)**
(30) Points A and D are on the same side of line $\overleftrightarrow{BG}$. **(Justify.)**
(31) Point D is in the interior of $\angle ABG$. **(Justify.)**
(32) $m(\angle ABD) + m(\angle DBG) = m(\angle ABG)$. **(Justify.)**
(33) $x + r = m(\angle ABG)$. **(Justify.)**
(34) $x + (180 - x) = m(\angle ABG)$. **(Justify.)**
(35) $180 = m(\angle ABG)$. **(Justify.)**
(36) Statement (36) is a contradiction. (What does it contradict?) Therefore, our assumption in step (20) was incorrect. That is, $x + y$ cannot be greater than 180.

Conclusion
(37) Conclude that $x + y = 180$.

End of Proof

Now that you see how difficult the proof of the Linear Pair Theorem is, you can understand why a typical high school book would simply include the statement as an axiom. We were introduced to the SMSG axioms back in Section 3.10 Distance and Rulers in High School Geometry Books (on page 91) Here is the SMSG axiom about linear pairs, an axiom used in many high school books:

SMSG Postulate 14: (Supplement Postulate) If two angles form a linear pair, then they are supplementary.

There is a converse for the Linear Pair Theorem. The converse is stated as here as a theorem but is not proven. You will be asked to prove it in a homework exercise.

Theorem 42 Converse of the Linear Pair Theorem

If adjacent angles have measures whose sum is 180, then the angles form a linear pair. That is, if angles $\angle ABD$ and $\angle DBC$ are adjacent and $m(\angle ABD) + m(\angle DBC) = 180$, then $A * B * C$.

Proof (for readers interested in advanced topics and for graduate students)

The proof is left to the reader.

The concept of a vertical pair of angles is very simple. Here is the definition.

Definition 45 vertical pair

A vertical pair is a pair of angles with the property that the sides of one angle are the opposite rays of the sides of the other angle.

It is a shame that such a non-descriptive name has become standard for the type of angles just described. It would make so much more sense to call them an "opposite pair"! But even with the non-descriptive name for the angles, everybody seems to remember the following theorem about them.

Theorem 43 Vertical Pair Theorem

If two angles form a vertical pair then they have the same measure.

Proof

(1) Suppose that two angles form a vertical pair.

(2) The angles can be labeled $\angle ABC$ and $\angle DBE$, where $A * B * D$ and $C * B * E$.

Fill in the missing steps, with justifications.

(*) $m(\angle ABC) = m(\angle DBE)$. **(Justify.)**

End of Proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 6.8 on page 153.

6.5. A digression about terminology

One of the difficult things about axiomatic geometry is the sheer number of definitions involved. To some extent, this is unavoidable because there are many configurations of objects possible, and many observations that can be made about them. If we described every configuration and every observation using only the primitive terms (*point*, *line*, and *lies on*), our writing would become so bloated that it would be unreadable. So we introduce defined objects and defined terms to streamline our writing. But it is also a fact that there are more geometry terms in common usage than are actually necessary in any particular version of geometry.

That last sentence is vague, so let me give an example. In this book, we have defined *congruent segments* to mean segments that have the *same length*. Within this book, the use of the expression *congruent segments* is completely unnecessary and adds a layer of notational hassle that can only cloud our understanding. Everytime you read something like "segment $\overline{AB}$ is congruent to segment $\overline{CD}$", you have to remind yourself to perform a little internal translation and register the true meaning of the phrase, "segments $\overline{AB}$ and $\overline{CD}$ have the same length". There is no question that this book would be clearer if we just omitted the definition of *segment congruence* and always referred to *segments with the same length* instead.

So why don't we do that? Why don't all books do that? Let me start by answering the second question. In some axiomatic geometry books, there are no axioms about distance and coordinate functions. Think about what that implies. If there is no distance function, one cannot define the length of a line segment in terms of distance. If there are no coordinate functions, then one cannot define betweenness of points in terms of their coordinates. In some such books, one finds axioms about betweenness of points and congruence of segments, where *between* and *congruent* are primitive, undefined terms. So the answer to the second question is that some books use terms like *congruent segments* because those books are presenting a different version of axiomatic geometry.

For example, Hilbert's axioms for Euclidean geometry are an axiom system in which *between* and *congruent* are primitive, undefined terms. One of his axioms is the following:

Hilbert's Congruence Axiom 3. If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.

That axiom uses only the primitive terms *point*, *between*, and *congruent segments* (abbreviated in symbols) and the logical connectives *If-then* and *and*. There is no extra layer of definitions to be decoded, no fat that could be trimmed to make the writing clearer.

But what about the first question? If it would clarify *this* book to omit the definition of *segment congruence* and refer to *segments with the same length* instead, why not do it? The answer is that books should use as universal a language as possible, so that the material presented one book can be more easily compared to the material in others. For example, consider our Neutral Geometry Theorem 25.

Theorem 25 Congruent Segment Addition Theorem.
If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.

The words and symbols of our Theorem 25 exactly match the words and symbols of Hilbert's Congruence Axiom 3. In our book, the statement is a theorem, not an axiom, and the terms *between* and *congruent segments* are defined terms, not primitive.
But if it is sometimes worthwhile to keep a few more definitions in use than are absolutely necessary, it is also sometimes important to avoid certain definitions. The term *supplementary angles* is one that we will not use in this book. Even so, it is important for you to know about the term and to be aware of its uses in other books.

In the book Euclidean and Non-Euclidean Geometries by Greenberg, the term *supplementary angles* is defined the way that I define *linear pair* in this book. In Greenberg's book, a theorem proves that if two angles are supplementary, then the sum of their measures is 180.

In the book Elementary Geometry from an Advanced Standpoint by Moise, the expression *linear pair* is defined in the same way that I define *linear pair* in this book. In Moise's book, the expression *supplementary angles* is defined to mean angles whose measures add up to 180. A theorem proves that if two angles form a linear pair, then they are supplementary.

In my book, I have chosen to use the term *linear pair* because the words have something to do with how the angles actually look. I will avoid all use of the expression *supplementary angles* because the expression is unnecessary and has inconsistent usages in the literature. If I want to talk about angles whose measures add up to 180, I will refer to them that way, rather than using some defined term.

Enough of this digression. Let's move on to study some new things.

6.6. Right Angles and Perpendicular Lines

Before defining right angles, it is useful to state and prove the following theorem.

Theorem 44 about angles with measure 90
For any angle, the following two statements are equivalent.
(i) There exists another angle that forms a linear pair with the given angle and that has the same measure.
(ii) The given angle has measure 90.

Proof
(1) Suppose that $\angle ABC$ is given.
Proof that (i) ➔ (ii)
(2) Suppose that statement (i) is true. That is, suppose that there exists an angle $\angle CBD$ such that the two angles form a linear pair and such that $m(\angle ABC) = m(\angle CBD)$.

(Fill in the missing steps.)

(*) $m(\angle ABC) = 90$. That is, statement (ii) is true.
Proof that (ii) ➔ (i)
(*) Suppose that statement (ii) is true. That is, suppose that $m(\angle ABC) = 90$.
(*) There exists a point D such that $A * B * D$. **(Justify.)**

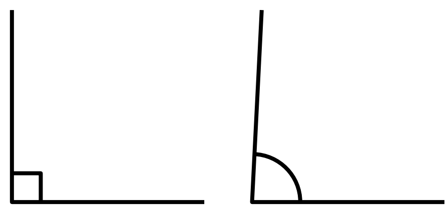
(Fill in the missing steps.)

(*) $m(\angle CBD) = 90$. **(Justify.)**
(*) Summarizing, the two angles $\angle ABC$ and $\angle CBD$ form a linear pair and have the same measure. That is, statement (i) is true.

End of Proof

The above theorem tells us that there is something special about angles with measure 90. We give them a special name, and we define some related terms at the same time.

Definition 46 acute angle, right angle, obtuse angle
An acute angle is an angle with measure less than 90.
A right angle is an angle with measure 90.
An obtuse angle is an angle with measure greater than 90.
In drawings, right angles are indicated by little boxes at the vertex.



With the definition of right angle comes some the following related definitions of perpendicular lines, segments, and rays.

Definition 47 perpendicular lines
Two lines are said to be perpendicular if there exist two rays that lie in the lines and whose union is a right angle. The symbol $L \perp M$ is used to denote that lines L and M are perpendicular.

Note that the previous definition is very terse: it does not say much about the lines and rays. Even so, we can infer more than was said. Here are two observations.

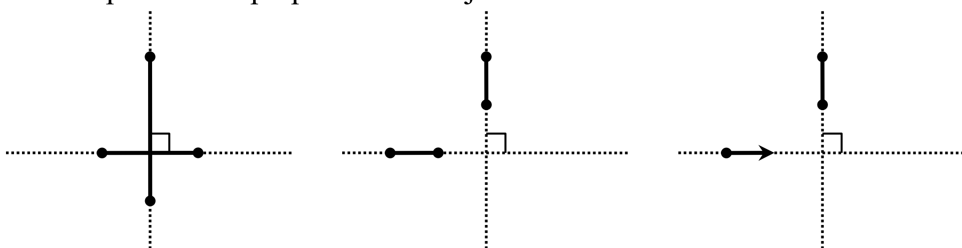
- The two rays cannot be collinear, because collinear rays do not form an angle. Therefore, one ray must lie in one line, and the other ray must lie in the other line, and the two lines cannot be the same line.
- The rays must share an endpoint in order to form an angle. This tells us that the two lines must intersect.

It will be useful to have a definition of perpendicular segments and perpendicular rays. We should consider perpendicular drawn objects in order to get an idea of how the definition should be worded for the corresponding abstract objects. You can certainly visualize perpendicular lines, so I do not need to draw a picture. And if I ask you to visualize perpendicular *segments*, or perpendicular *rays*, you will probably think of a drawing where two segments or two rays intersect at right angles. But it is very useful to define perpendicularity for segments and rays in a way that it does not require them to intersect. Here is the definition.

Definition 48 perpendicular lines, segments, rays
Suppose that Object 1 is a line or a segment or a ray and that Object 2 is a line or a segment or a ray. Object 1 is said to be perpendicular to Object 2 if the line that contains Object 1 is

perpendicular to the line that contains Object 2 by the definition of perpendicular lines in the previous definition. The symbol $L \perp M$ is used to denote that objects L and M are perpendicular.

Here are some pictures of perpendicular objects



perpendicular segments | perpendicular segments | segment perpendicular to ray

The definition just presented essentially expanded our definition of perpendicular to include more cases. The following theorem reminds us that at any intersection of perpendicular lines, four perpendicular angles are formed.

Theorem 45 If two intersecting lines form a right angle, then they actually form four.

Proof

(1) Suppose two intersecting lines form a right angle. That is, line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{AC}$ have the property that $m(\angle BAC) = 90$.

(2) There exists a point D such that $D * A * B$ and a point E such that $E * A * C$. **(Justify.)**

Fill in the remaining steps.

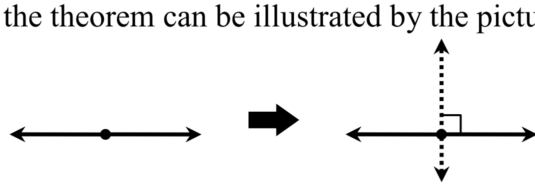
End of Proof

In the previous section, we studied Theorem 40 about the existence and uniqueness of angle bisectors. For such a simple-sounding theorem, the proof was surprisingly difficult, and used <N8> The Angle Construction Axiom. We will now study another simple sounding yet very important theorem whose proof also uses Axiom <N8>. This time, the proof will not be so hard.

Theorem 46 existence and uniqueness of a line that is perpendicular to a given line through a given point that *lies on* the given line

For any given line, and any given point that *lies on* the given line, there is exactly one line that passes through the given point and is perpendicular to the given line.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 46

Proof

(1) Suppose that L is a line and P is a point that lies on L. **(Make a drawing.)**

Part 1: Show that a line exists that passes through P and is perpendicular to L.

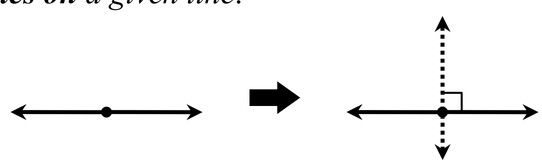
(2) There exists a second point on L. **(Justify.)** Call the second point Q. **(Make a new drawing.)**
(3) There exists a point that is not on L. **(Justify.)** Call it R. **(Make a new drawing.)**
(4) Point R lies in one of the half-planes created by line L. **(Justify.)** Let H_R be that half-plane.
(5) There exists a ray $\overrightarrow{PS}$ such that $S \in H_R$ and $m(\angle QPS) = 90$. **(Justify.)**
(6) Let M be line $\overleftrightarrow{PS}$. Observe that M passes through P and $M \perp L$.

Part 2: Show that the line is unique.

(7) Suppose that some line N passes through P and that $N \perp L$. **(Make a new drawing.)**
(8) There exist points T and U on line N such that $T * P * U$. **(Justify.)**
(9) Points T and U lie on opposite sides of line L, so exactly one of them lies in half-plane H_R. Without loss of generality (WLOG), we may assume that it is point T that lies in half-plane H_R. (If the assumption is not true, we can simply interchange the names of points T and U so that the assumption *is* true.) **(Make a new drawing.)**
(10) Note that ray $\overrightarrow{PT}$ has the property that point T lies in half-plane H_R and $m(\angle QPT) = 90$.
(11) Ray $\overrightarrow{PT}$ must be the same as ray $\overrightarrow{PS}$. **(Justify.)** So point T must lie on ray $\overrightarrow{PS}$ and hence also on line M. Therefore, line N is the same as line M.

End of Proof

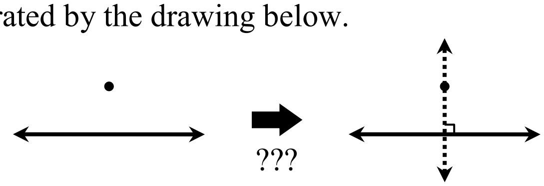
Notice that Theorem 46 is about the existence and uniqueness of a line perpendicular to a line through *a given point that* ***lies on*** *a given line*.



Theorem 46

The proof made use of earlier theorems that relied on axioms <N2> through <N8>.

Consider the different situation where there is a given line and *a given point that* ***does not lie on*** *the given line*. Notice that Theorem 46 does not apply to this situation. It is natural to wonder if there exists a unique line that passes through the given point and is perpendicular to the given line. This question is illustrated by the drawing below.



It turns out that there does exist a unique line that passes through the given point and is perpendicular to the given line, but the proof requires theorems that follow from axiom <N10>. We will not be studying that axiom until the next chapter. So we will have to wait until the next chapter for the theorem about the existence and uniqueness of a line perpendicular to a given line through a *given point that* ***does not lie on*** *the line*.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 6.8 on page 153.

6.7. Angle Congruence

In Definition 29, segment congruence, found on page104, we defined congruent segments to be segments that have the same length. In Section 6.5 (A digression about terminology), I pointed out that we did not really need to use the term *congruent segments*: we could just as easily refer to *segments of equal length*. The reason for using the term *congruent segments* was that it kept the language of this book more in line with the language of other books. We will define *congruent angles* in an analogous way.

Definition 49 angle congruence

Two angles are said to be *congruent* if they have the same measure. The symbol ≅ is used to indicate this. For example $\angle ABC \cong \angle DEF$ means $m(\angle ABC) = m(\angle DEF)$.

I have postponed making this definition until now, at the end of this chapter about angle measure, because I wanted you to see that we really do not have to use the term *congruent angles* in this book. Throughout this chapter, it has not been an inconvenience to refer to *angles of equal measure*. But as with *segment congruence*, one should not feel compelled to avoid the terminology of *angle congruence*. It is too widely-used to be avoided, and there are settings where it *must* be used. An example of a setting would be Hilbert's Axioms for Euclidean Geometry, where there is no angle measurement axiom, and *angle congruence* is a primitive, undefined term.

The four theorems below would be necessary if we were using an axiom system where *angle congruence* is a primitive, undefined term. They are not necessary when using axiom systems such as ours, axiom systems that have an angle measurement axiom. But some books that use axiom systems like ours include these theorems, anyway. I include them here, and will assign their proofs to you as exercises, so that you will be familiar with them.

Theorem 47 Angle congruence is an equivalence relation.

Theorem 48 Congruent Angle Construction Theorem

Let $\overrightarrow{AB}$ be a ray on the edge of a half-plane H. For any angle $\angle CDE$, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $\angle PAB \cong \angle CDE$.

Theorem 49 Congruent Angle Addition Theorem

If point D lies in the interior of $\angle ABC$ and point D' lies in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle DBC \cong \angle D'B'C'$, then $\angle ABC \cong \angle A'B'C'$.

Theorem 50 Congruent Angle Subtraction Theorem

If point D lies in the interior of $\angle ABC$ and point D' lies in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle ABC \cong \angle A'B'C'$, then $\angle DBC \cong \angle D'B'C'$.

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6.8. Exercises for Chapter 6

Exercises for Section 6.1 The Angle Measurement Axiom

[1] Are there straight angles in our Neutral Geometry? Explain.
[2] Can an angle have measure 180 in our Neutral Geometry? Explain?
[3] Can an angle have measure 0 in our Neutral Geometry? Explain.

[4] In our Neutral Geometry, if $m(\angle ABC) = 1.37$, does that mean 1.37 degrees, or 1.37 gradians, or 1.37 radians? Explain.

[5] In our Neutral Geometry, why do angle measurements not get labeled as degrees, gradians, or radians? Explain.

[6] Here is a statement and justification from a proof written by Waldo. What's wrong with it?
(17) Angle $\angle DEF$ has measure $m(\angle DEF) = 90$. (By Angle Measurement Axiom <N7>)

Exercises for Section 6.3 The Angle Measure Addition Axiom

[7] (advanced) Draw diagrams to illustrate and justify the steps in the proof of Theorem 39 (about points in the interior of angles) found on page 141.

[8] Draw diagrams to illustrate and justify the steps in the proof of Theorem 40 (Every angle has a unique bisector.), found on page 143.

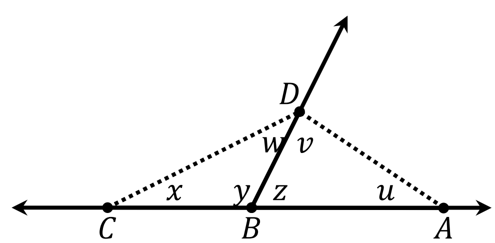
Exercises for Section 6.4 The Linear Pair Theorem

[9] (advanced) Draw diagrams to illustrate and justify the steps in the proof of Theorem 41 (Linear Pair Theorem.) found on page 144.

[10] Here is an idea for a short proof of Theorem 41 (Linear Pair Theorem.). It much shorter than the proof on page 144.

Idea for a Proof of the Linear Pair Theorem

(1) Suppose that two angles form a linear pair.
(2) The angles can be labeled $\angle ABD$ and $\angle DBC$ where $A * B * C$.
(3) Consider the angles of triangles ΔABD and ΔDBC and ΔADC as shown in the diagram at right. The six letters u, v, w, x, y, z stand for the measures of the six angles shown.



(4) $u + v + z = 180$ (because the measures of the three angles of any triangle add up to 180).
(5) $w + x + y = 180$ (for the same reason).
(6) Therefore, $u + v + w + x + y + z = 360$.
(7) But $u + v + w + x = 180$ (because the measures of the three angles of triangle ΔADC add up to 180).
(8) Therefore, $y + z = 180$.

End of Proof

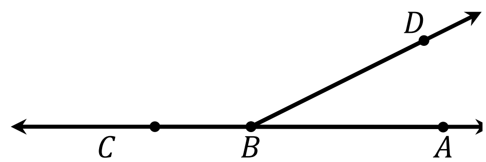
Is this a valid proof? Can it be used as the proof of our Theorem 41? Explain.

[11] Theorem 41 (Linear Pair Theorem.), found on page 144, states that if two angles form a linear pair, then the sum of the measures of their angles is 180. An attempt at a proof is shown below. For statements (2), (3), (4), either justify the statement or explain why the statement is invalid.

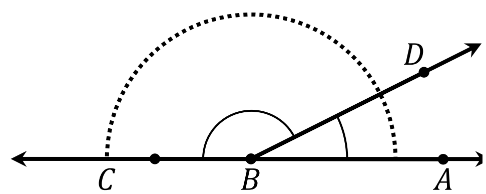
Proof

(1) Suppose that two angles form a linear pair.

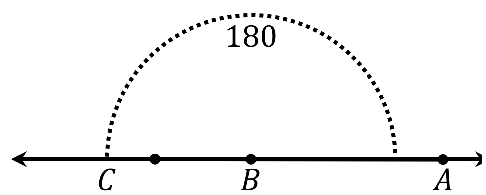
(2)The angles can be labeled $\angle ABD$ and $\angle DBC$ with $A * B * C$.
Justify or explain why the statement is invalid: ________



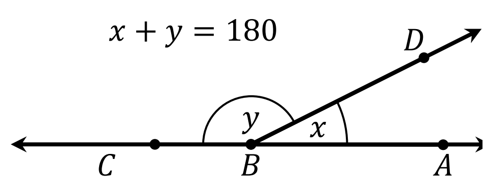
(3) $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$
Justify or explain why the statement is invalid: ________



(4) $m(\angle ABC) = 180$
Justify or explain why the statement is invalid: ________



(5) Therefore, $m(\angle ABD) + m(\angle DBC) = 180$ (by statements (3) and (4) and transitivity).



End of Proof

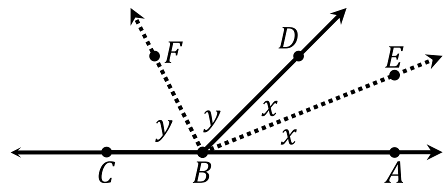
[12] (advanced) Prove Theorem 42 (Converse of the Linear Pair Theorem), found on page 146. Hint: The goal is to show that $A * B * C$. Show that there exists a point E such that $A * B * E$. Show that $m(\angle ABD) + m(\angle DBE) = 180$. Then find a way to show that rays $\overrightarrow{BC}$ and $\overrightarrow{BE}$ must be the same ray. Use that to reach a conclusion.

[13] Fill in the missing steps and justifications for the proof of Theorem 43 (Vertical Pair Theorem), on page 146. Write up the complete proof with numbered steps. Make drawings.

Exercises for Section 6.6 Right Angles and Perpendicular Lines

[14] Fill in the missing steps and justifications for the proof of Theorem 44 (about angles with measure 90), found on page 148. Write up the complete proof with numbered step and drawings.

[15] In the figure at right,
Points A, B, C are collinear.
Ray $\overrightarrow{BE}$ bisects angle $\angle ABD$.
Ray $\overrightarrow{BF}$ bisects angle $\angle DBC$.
Prove that angle $\angle EBF$ is a right angle.



Hint: Let $x = m(\angle ABE) = m(\angle EBD)$ and let $y = m(\angle DBF) = m(\angle FBC)$.

[16] Fill in the missing steps and justifications for the proof of Theorem 45 (If two intersecting lines form a right angle, then they actually form four.), found on page 150. Write up the complete proof with numbered steps. Make drawings to illustrate.

[17] Draw diagrams to illustrate and justify the steps in the proof of Theorem 46 (existence and uniqueness of a line that is perpendicular to a given line through a given point that *lies on* the given line), found on page 150.

Exercises for Section 6.7 Angle Congruence

[18] Prove Theorem 47 (Angle congruence is an equivalence relation.), found on page 152.

[19] Prove Theorem 48 (Congruent Angle Construction Theorem), found on page 152. Hints:
- Think about proof structure. All the given information should go in the first step.
- Use the angle measurement axiom to state that there is a number $r = m(\angle CDE)$.
- Then use the angle construction axiom.
- Then use the definition of congruence.

[20] Prove Theorem 49 (Congruent Angle Addition Theorem), found on page 152. Hints:
- Think about proof structure. All the given information should go in the first step.
- Use the angle measurement axiom to state that there are numbers that are measures of the angles involved. (Give the numbers letter names.)
- Use the definition of congruence to tell you about some equalities among the numbers that you have introduced
- Then use the angle measure addition axiom to find out more about the numbers.
- Then use the definition of congruence again.

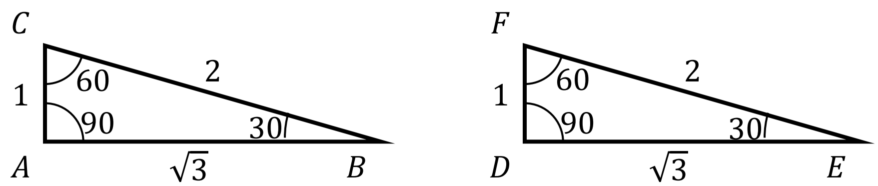
.

7. Neutral Geometry V: The Axiom of Triangle Congruence

In this chapter we will be introduced to the notion of triangle congruence and to the last axiom of Neutral Geometry. Before starting that project, it is a good idea to review the meaning of equality of sets.

Definition 50 symbol for *equality* of two sets
Words: $S\ equals\ T$.
Symbol: $S = T$.
Usage: S and T are sets.
Meaning: S and T are the same set. That is, every element of set S is also an element of set T, and vice-versa.

It is worthwhile to consider a few examples. Refer to the drawing below.



(1) The statement "$\Delta ABC = \Delta ABC$" is true because the two sets are clearly the same.

(2) The statement "$\Delta ABC = \Delta ACB$" is true. To see why, observe that
$$\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA} = \overline{BA} \cup \overline{CB} \cup \overline{AC} = \overline{AC} \cup \overline{CB} \cup \overline{BA} = \Delta ACB$$
(3) The statement "$\Delta ABC = \Delta DEF$" is false because the sets are not the same. Point D is in set ΔDEF but it is not in set ΔABC.

In this section, we will learn about triangle congruence. We will learn that triangles ΔABC and ΔDEF in the drawing above are congruent, and we will learn that the symbol $\Delta ABC \cong \Delta DEF$ is appropriate. But that is not the same thing as saying that the triangles ΔABC and ΔDEF are equal. We have seen that they are not equal.

7.1. The Concept of Triangle Congruence

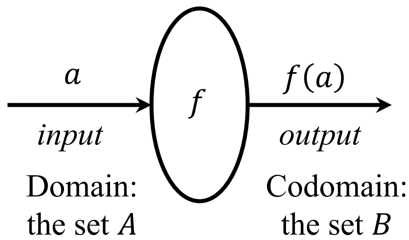
7.1.1. Correspondences between parts of triangles

The term *correspondence* is used in almost any discussion of triangle congruence, and in almost any theorem about triangle congruence, so we start our presentation of the concept of triangle congruence by defining the term.

Definition 51 function, domain, codomain, image, machine diagram, correspondence
Symbol: $f: A \to B$
Spoken: "f is a function that maps A to B."
Usage: A and B are sets. Set A is called the *domain* and set B is called the *codomain*.
Meaning: f is a machine that takes an element of set A as input and produces an element of set B as output.

More notation: If an element $a \in A$ is used as the input to the function , then the symbol $f(a)$ is used to denote the corresponding output. The output $f(a)$ is called *the image of* a *under the map* f.

Machine Diagram:



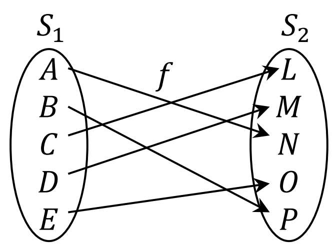
Additional notation: If f is both one-to-one and onto (that is, if f is a bijection), then the symbol $f: A \leftrightarrow B$ will be used. In this case, f is called a *correspondence* between the sets A and B.

Correspondences play a key role in the concepts of triangle similarity and congruence, and they will also play a key role in the concepts of polygon similarity and congruence, so we should do a few examples to get more familiar with them.

Examples

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be the cubing function, $f(x) = x^3$. Then f is one-to-one and onto, so we could say that f is a *correspondence*, and we would write $f: \mathbb{R} \leftrightarrow \mathbb{R}$.

(2) Let $S_1 = \{A, B, C, D, E\}$ and $S_2 = \{L, M, N, O, P\}$. Define a function $f: S_1 \to S_2$ by this picture:



Then we would say that f is a correspondence, and we would write $f: S_1 \leftrightarrow S_2$. It would be appropriate to replace all of the arrows in the diagram with double arrows, $\leftrightarrow$.

(3) For the same example as above, we could display the correspondence more concisely:

$$
\begin{aligned}
A &\leftrightarrow N \\
B &\leftrightarrow P \\
C &\leftrightarrow L \\
D &\leftrightarrow M \\
E &\leftrightarrow O
\end{aligned}
$$

This takes up much less space, and is faster to write, than the picture. However, notice that this way of displaying the correspondence still uses a lot of space.

(4) There is an even more concise way to display the correspondence from the above example. To understand the notation, though, we should first recall some conventions about brackets and parentheses. When displaying sets, curly brackets are used. In sets, order is not important. For example, $S_1 = \{A, B, C, D, E\} = \{C, A, E, D, B\}$. When displaying an *ordered* list, parentheses are used. So whereas the $\{A, B\}$ and $\{B, A\}$ are the same set, (A, B) and (B, A) are different ordered pairs. With that notation in mind, we will use the symbol below to denote the function f described in the previous examples.

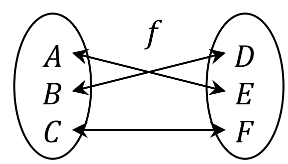
$$(A, B, C, D, E) \leftrightarrow (N, P, L, M, O)$$

The parentheses indicate that the order of the elements is important, and the double arrow symbol indicates that there is a correspondence between the lists. Notice that this way of displaying the function is not as clear as the one in the previous example, but it takes up much less space.

Definition 52 Correspondence between vertices of two triangles
Words: "f is a correspondence between the vertices of triangles ΔABC and ΔDEF."
Meaning: f is a one-to-one, onto function with domain $\{A, B, C\}$ and codomain $\{D, E, F\}$.

Examples of correspondences between the vertices of triangles ΔABC and ΔDEF.
(1) $(A, B, C) \leftrightarrow (D, E, F)$
(2) $(A, B, C) \leftrightarrow (D, F, E)$
(3) $(B, A, C) \leftrightarrow (D, E, F)$
(4) $(B, C, A) \leftrightarrow (D, F, E)$

Notice that the third and fourth examples are actually the same. Each could be illustrated by the figure shown at right.



If a correspondence between the vertices of two triangles has been given, then there is an automatic correspondence between any other geometric items that are defined purely in terms of those vertices. For example, suppose that we are given the correspondence $(A, B, C) \leftrightarrow (D, E, F)$ between the vertices of ΔABC and ΔDEF. There is a correspondence between the sides of triangle ΔABC and the sides of ΔDEF, and a correspondence between the angles of triangle ΔABC and the angles of ΔDEF, since those items are defined only in terms of the vertices. For clarity, we can display all correspondences in a vertical list.

$A \leftrightarrow D$ $B \leftrightarrow E$ $C \leftrightarrow F$	given correspondence between vertices of ΔABC and vertices of ΔDEF.
$\overline{AB} \leftrightarrow \overline{DE}$ $\overline{BC} \leftrightarrow \overline{EF}$ $\overline{CA} \leftrightarrow \overline{FD}$ $\angle ABC \leftrightarrow \angle DEF$ $\angle BCA \leftrightarrow \angle EFD$ $\angle CAB \leftrightarrow \angle FDE$	automatic correspondence between parts of ΔABC and parts of ΔDEF.

Based on the ideas of this discussion, we make the following definition.

Definition 53 corresponding parts of two triangles

Words: Corresponding parts of triangles ΔABC and ΔDEF.

Usage: A correspondence between the vertices of triangles ΔABC and ΔDEF has been given.

Meaning: As discussed above, if a correspondence between the vertices of triangles ΔABC and ΔDEF has been given, then there is an automatic correspondence between the sides of triangle ΔABC and and the sides of triangle ΔDEF, and also between the angles of triangle ΔABC and the angles of ΔDEF, For example, if the correspondence between vertices were $(A, B, C) \leftrightarrow (D, E, F)$, then corresponding parts would be pairs such as the pair of sides $\overline{AB} \leftrightarrow \overline{DE}$ and the pair of angles $\angle ABC \leftrightarrow \angle DEF$.

7.1.2. Definition of Triangle Congruence

Now that we have a clear understanding of what is meant by the phrase "corresponding parts" of two triangles, we can make a concise definition of *triangle congruence*.

Definition 54 triangle congruence

To say that two triangles are *congruent* means that there exists a correspondence between the vertices of the two triangles such that corresponding parts of the two triangles are congruent. If a correspondence between vertices of two triangles has the property that corresponding parts are congruent, then the correspondence is called *a congruence.* That is, the expression *a congruence* refers to a particular correspondence of vertices that has the special property that corresponding parts of the triangles are congruent.

Remark: Many students remember the sentence "Corresponding parts of congruent triangles are congruent" from their high school geometry course. The acronym is, of course, "*CPCTC*". We see now that in this book, "*CPCTC*" is really a summary of the definition of triangle congruence. That is, to say that two triangles are congruent is the same as saying that corresponding parts of those two triangles are congruent. This is worth restating: In this book, *CPCTC* is not an axiom and it is not a theorem; it is merely an acronym for the definition of triangle congruence.

An important fact about triangle congruence is stated in the following theorem. You will be asked to prove the theorem in the exercises.

Theorem 51 triangle congruence is an equivalence relation

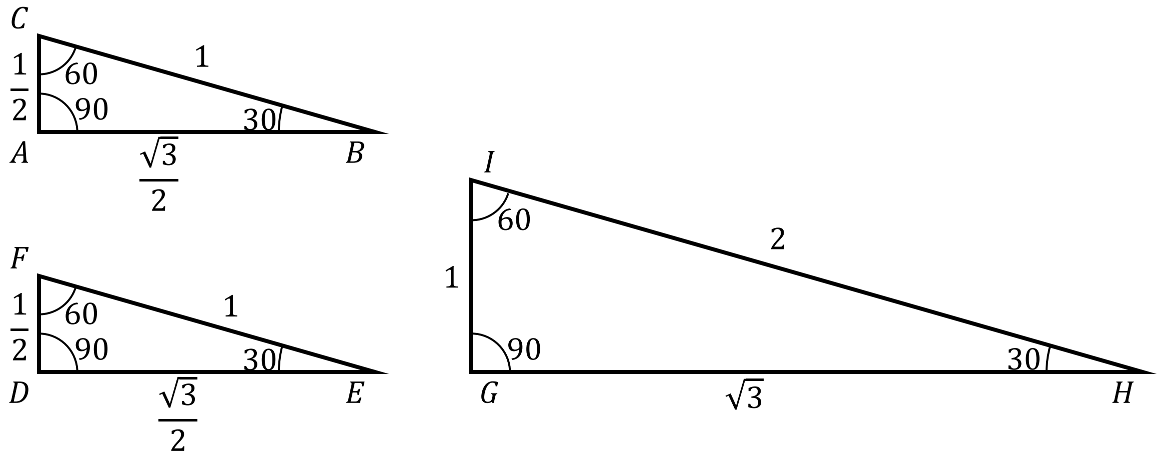
It is important to discuss notation at this point. It is no accident that Definition 54 above does not include a symbol. There is no commonly-used symbol whose meaning matches the definition of triangle congruence. This may surprise you, because you have all seen the symbol $\cong$ put between triangles. But that symbol means something different, and the difference is subtle. Here is the definition.

Definition 55 symbol for *a congruence* of two triangles

Symbol: $\Delta ABC \cong \Delta DEF$.

Meaning: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ of vertices is a congruence.

You should be a little confused. There is more subtlety in the notation than you might have realized. It is worthwhile to consider a few examples. Refer to the drawing below.



Easy examples involving ΔABC and ΔDEF.

- The statement "ΔABC is congruent to ΔDEF" is true. Proof: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a congruence.
- The statement "ΔABC is congruent to ΔDFE" is true. Proof: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a congruence. This is the same correspondence from the previous example. Since there exists a correspondence that is a congruence, we say that the triangles are congruent.
- The statement "$\Delta ABC \cong \Delta DEF$" is true, because the correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a congruence.
- The statement "$\Delta ABC \cong \Delta DFE$" is false, because the correspondence $(A, B, C) \leftrightarrow (D, F, E)$ is not a congruence. Observe that $m(\angle ABC) = 30$ while the corresponding angle has measure $m(\angle DFE) = 60$.

More subtle examples involving ΔABC and ΔABC.

- The statement "ΔABC is congruent to ΔABC" is true. Proof: The correspondence $(A, B, C) \leftrightarrow (A, B, C)$ is a congruence.
- The statement "ΔABC is congruent to ΔACB" is true. Proof: The correspondence $(A, B, C) \leftrightarrow (A, B, C)$ is a congruence. This is the same correspondence from the previous example. Since there exists a correspondence that is a congruence, we say that the triangles are congruent.
- The statement "$\Delta ABC \cong \Delta ABC$" is true, because the correspondence $(A, B, C) \leftrightarrow (A, B, C)$ is a congruence.
- The statement "$\Delta ABC \cong \Delta ACB$" is false, because the correspondence $(A, B, C) \leftrightarrow (A, C, B)$ is not a congruence. Observe that $m(\angle ABC) = 30$ while the corresponding angle has measure $m(\angle ACB) = 60$.

Examples involving ΔABC and ΔGHI.

- The statement "ΔABC is congruent to ΔGHI" is false. There is no way to define a correspondence of vertices such that all corresponding parts are congruent.
- The statement "$\Delta ABC \cong \Delta GHI$" is false, because the correspondence $(A, B, C) \leftrightarrow (G, H, I)$ is not a congruence. Observe that $length(\overline{BC}) = 1$ while the corresponding side has $length(\overline{HI}) = 2$.

7.1.3. The Axiom of Triangle Congruence

When comparing drawn triangles, one of the things that we can do is slide one drawn triangle on top of another and seeing if they fit. In our drawings, we know that if enough parts of one drawing fit on top of the corresponding parts of another drawing, then all of the other parts will fit, as well.

This can be said more precisely. To determine whether or not two drawn triangles fit on top of each other, one would officially have to verify that every pair of corresponding drawn line segments fit on top of each other and also that every pair of corresponding drawn angles fit on top of each other. That is total of six fits that must be checked. But we know that with drawings, one does not really need to check all six fits. Certain combinations of three fits are enough. Here are four examples of sets of three fits that will guarantee that two drawn triangles will fit perfectly on top of each other:

(1) If two sides and the included angle of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.

(2) If two angles and the included side of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.

(3) If all three sides of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.

(4) If two angles and some non-included side of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.

We would like our abstract line segment congruence and angle congruence and triangle congruence to have this same sort of behavior. But if we want them to have that behavior, we must specify it in the Neutral Geometry axioms. One might think that it would be necessary to include four axioms, to guarantee that the four kinds of behavior that we observe in drawn triangles will also be observed in abstract triangles. But the amazing thing is that we don't need to include four axioms. We can include just one axiom, about just one kind of behavior that we want abstract triangles to have, and then prove theorems that show that triangles will also have the other three kinds of desired behavior.

Here is the axiom that will be included, along with the three theorems that will be presented and proven in the remainder of this chapter.

Axiom <N10> (*SAS* Axiom): If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

The *ASA* Congruence Theorem: In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

The *SSS* Congruence Theorem: In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

The *AAS* Congruence Theorem: In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Note that the statements of the three congruence theorems have been mentioned here just as an introduction to the coming material. The three theorems have not yet been proven and they do not yet have theorem numbers, so we may not yet use any of them in proofs. Soon, but not yet.

7.1.4. Digression about the names of theorems

A digression about the names of theorems. So far in this book, I have tagged each theorem with some sort of description that I made up. I will continue that practice in the coming chapters. But from now on, many of our theorems will have names that are widely used, and I will present those names as well. Many of the names follow an informal convention for naming theorems: They are often named for the situation described in their hypotheses.

For example, suppose two theorems are stated as follows.

- Theorem 1: If the dog is blue, then the car is red.
- Theorem 2: If the car is red, then the bear is hungry.

Following the naming convention, Theorem 1 would be called "The Blue Dog Theorem", and Theorem 2 would be called "The Red Car Theorem".

It is important to realize that "The Red Car Theorem" could never be used to prove that a car is red! The Red Car Theorem tells us something about the situation in which we already know that the car is red. (The theorem tells us that in that situation, the bear is hungry.) If we do not know that the car is red, and we want to prove that the car is red, then we will need a theorem that has the statement "the car is red" as part of the conclusion. We see that the Blue Dog Theorem would work. So one strategy for proving that the car is red would be to first prove somehow that the dog is blue, and then use the Blue Dog Theorem to prove that the car is red.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 7.8 on page 189.

7.2. Theorems about Congruences in Triangles

In Section 7.1.3, we made note of four familiar behaviors of drawn triangles that we expect to also be manifest in the abstract triangles of our axiomatic geometry. In that section, we discussed that only one of those behaviors needed to be specified in the list of axioms (Axiom <N10>, the *SAS* Congruence Axiom); the remaining three behaviors could then be proven as theorems. In this section, we will prove theorems about two of those three behaviors. We will prove the *ASA* Congruence Theorem and the *SSS* Congruence Theorem. (The fourth behavior will be proven in the next chapter, in the *AAS* Congruence Theorem.) Along the way, we will also prove a few other theorems about triangle behavior, theorems that will be needed for the proofs of the *ASA* and *SSS* Congruence Theorems and that will also be useful throughout the remainder of the book.

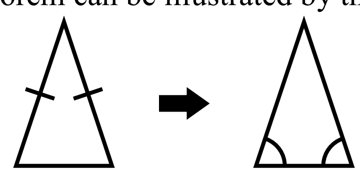
We start with a definition of some terminology pertaining to triangles.

Definition 56 scalene, isosceles, equilateral, equiangular triangles
A *scalene triangle* is one in which no two sides are congruent.
An *isosceles triangle* is one in which at least two sides are congruent.
An *equilateral triangle* is one in which all three sides are congruent.
An *equiangular triangle* is one in which all three angles are congruent.

Our first theorem is commonly called the Isosceles Triangle Congruence Theorem. Notice that the name of this theorem fits the naming convention. That is, the name of the theorem mentions isosceles triangle, and it is in the hypothesis of the theorem that isosceles triangle shows up, not in the conclusion. I prefer to refer to it as "the *CS* ➔ *CA* theorem for triangles" because that name encapsulates what the theorem actually says.

Theorem 52 the *CS* ➔ *CA* theorem for triangles (the Isosceles Triangle Theorem)
In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 52

Proof

(1) Suppose that ΔABC has $\overline{AB} \cong \overline{AC}$.

(2) Using the correspondence $(A, B, C) \leftrightarrow (A, C, B)$ between the vertices of ΔABC and ΔACB, we have the following pairs of corresponding parts

$$\begin{gathered} parts\ of\ \Delta ABC \leftrightarrow parts\ of\ \Delta ACB \\ \overline{AB} \leftrightarrow \overline{AC} \\ \angle CAB \leftrightarrow \angle BAC \\ \overline{AC} \leftrightarrow \overline{AB} \end{gathered}$$

Observe that the corresponding parts in each of these three pairs are congruent.

(3) The correspondence is a congruence. (In symbols, we would write $\Delta ABC \cong \Delta ACB$.) (by Axiom <N10>, the *SAS* congruence axiom) That is, all other pairs of corresponding parts are also congruent.

(4) Therefore, $\angle ABC \cong \angle ACB$.

End of proof

The following corollary is not difficult to prove. You will prove it in a homework exercise.

Theorem 53 (Corollary) In Neutral Geometry, if a triangle is equilateral then it is equiangular.

Recall that Theorem 40 states that every angle has a unique bisector. In order to prove it in the previous chapter, we had to first prove Theorem 39. Together, the proofs of those two theorems amounted to a rather difficult proof of the existence and uniqueness of angle bisectors. I promised that there would be an easier proof that used triangle congruence. With Neutral Axiom <N10> (*SAS* Congruence) and Theorem 52 (*CS* ➔ *CA*) now available as tools to prove theorems, an easier proof of the existence and uniqueness of angle bisectors is possible. You will be asked to justify the steps of an easier proof in the exercises.

Now we come to our first proof of one of the triangle behaviors mentioned in Section 7.1.3. I have supplied a drawing for each step. You will be asked to justify the steps in a class drill.

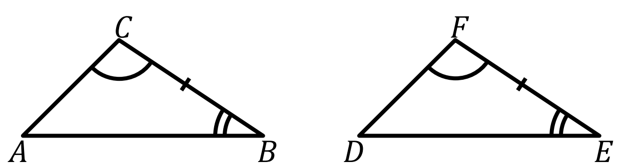
Theorem 54 the *ASA* Congruence Theorem for Neutral Geometry
In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Remark: The statement of the theorem can be illustrated by the picture below.

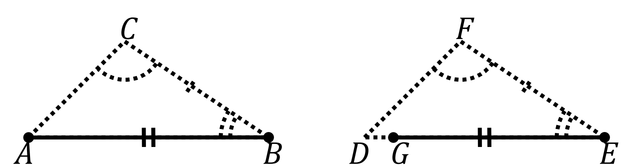
Theorem 54

Proof

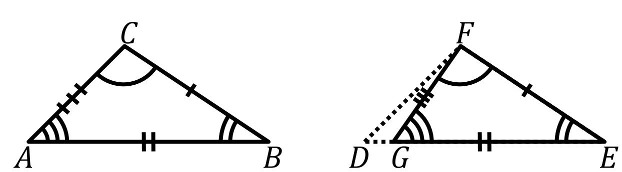
(1) Suppose that ΔABC and ΔDEF are given such that $\angle ABC \cong \angle DEF$ and $\overline{BC} \cong \overline{EF}$ and $\angle BCA \cong \angle EFD$.



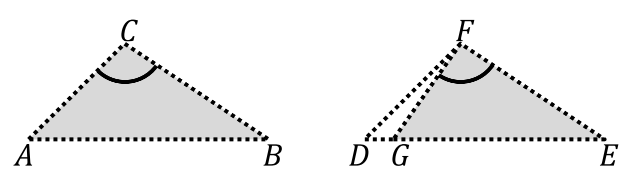
(2) There exists a point G on ray $\overrightarrow{ED}$ such that $\overline{EG} \cong \overline{BA}$. **(Justify.)** (We suspect that G is the same point as D, but we have not yet proven that, so we should not draw it that way.)



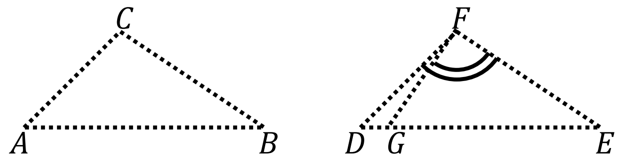
(3) $\Delta GEF \cong \Delta ABC$. **(Justify.)**



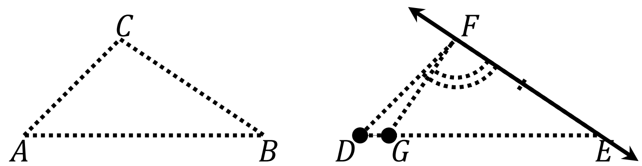
(4) $\angle EFG \cong \angle BCA$. **(Justify.)**



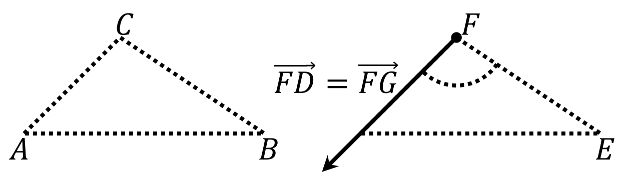
(5) $\angle EFG \cong \angle EFD$. **(Justify.)**



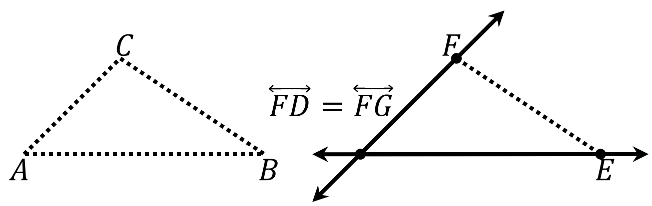
(6) Points D and G are on the same side of line $\overleftrightarrow{EF}$. **(Justify.)**



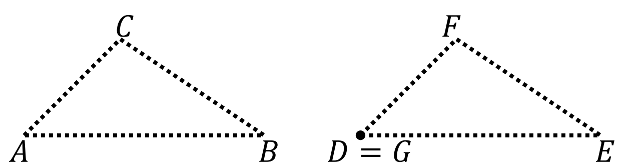
(7) Ray $\overrightarrow{FD}$ must be the same ray as $\overrightarrow{FG}$. **(Justify.)**



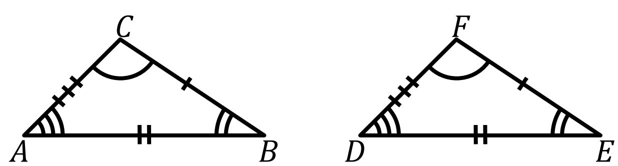
(8) Line $\overleftrightarrow{DF}$ can only intersect line $\overleftrightarrow{DE}$ at a single point. **(Justify.)**



(9) Points D, G must be the same point.



(10) $\Delta DEF \cong \Delta ABC$. **(Justify.)**



End of proof

•

7.2.1. Remarks About Drawings

Notice the way in which the drawings are presented in the proof above. Each step is accompanied by a drawing. But more than that, notice that each drawing is drawn in a way to focus attention on the objects that were mentioned in that step of the proof. For instance, step (2) mentions two congruent segments. In the drawing for that step, those segments are darker, with little marks on them to indicate that they are congruent. Other parts of the drawing are dotted, which de-emphasizes them.

Also recall the picture that accompanied the statement of Theorem 54. Observe that the picture is made up of two drawings. The drawing on the left (consisting of a pair of triangles with certain congruent parts indicated) illustrates the hypothesis, while the drawing on the right (consisting of two triangles with all corresponding parts congruent.) illustrates the conclusion.

Making drawings this way is clearly a lot of work, a lot more work than simply making a single drawing that includes all the objects mentioned in the theorem. But realize that the purpose of the drawings above was not to illustrate the *objects mentioned in the theorem*, but rather to illustrate the *statement of the theorem* or the *steps of the proof*. To illustrate the statement of a theorem, it really is helpful to have two drawings: one that illustrates the given information (or the hypothesis) and another one that illustrates the claim (or the conclusion). To illustrate the steps of a proof, it really is helpful to have a new drawing for each step.

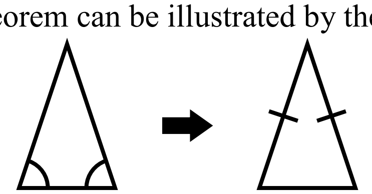
Sometimes you will be asked to provide a drawing that illustrates the statement of a theorem, and sometimes you will be asked to provide drawings that illustrate certain steps (or all of the steps) in the proof of a theorem. When I ask you to illustrate the statement of a theorem, I want you to make a picture that includes two drawings, one that illustrates the given information (or the hypothesis) and another one that illustrates the claim (or the conclusion). When I ask you to provide drawings that illustrate certain steps (or all of the steps), I really do mean for you to make lots of drawings, and to try to draw each in a way that emphasizes the information stated in the corresponding step. This can be tedious for you, but you will learn a lot.

7.2.2. Return to our discussion of Theorems about Triangles

Look back at the proof of Theorem 52 (the *CS* ➔ *CA* Theorem for Triangles). That proof used the *SAS* axiom and a trick involving a correspondence between the vertices of a single triangle. A similar trick can be used to prove the following theorem. This theorem is called the *CA* ➔ *CS* Theorem; its proof will use the same sort of trick involving a correspondence between the vertices of a single triangle, but the proof will cite the *ASA* Congruence Theorem, instead of the *SAS* axiom. You will write the proof for a homework exercise.

Theorem 55 the *CA* ➔ *CS* theorem for triangles in Neutral Geometry
In Neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if *CA* then *CS*.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 55

Here is an immediate corollary that is not difficult to prove. You will be asked to supply a proof in a homework exercise.

Theorem 56 (Corollary) In Neutral Geometry, if a triangle is equiangular then it is equilateral.

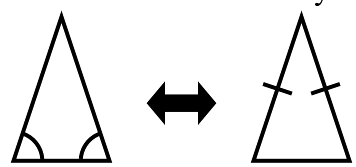
Consider what we know about congruent sides and congruent angles in triangles.

- From Theorem 52, we know that *CS* ➔ *CA*.
- From Theorem 55, we know that *CA* ➔ *CS*.

Combining, we have the following immediate corollary:

Theorem 57 (Corollary) The *CACS* theorem for triangles in Neutral Geometry.
In any triangle in Neutral Geometry, congruent angles are always opposite congruent sides.
That is, *CA* ⇔ *CS*.

Remark: The statement of the theorem can be illustrated by the picture below.

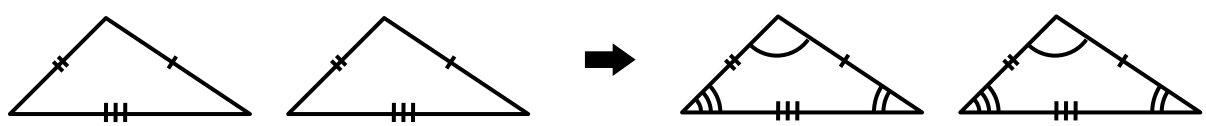


Theorem 57

So far in this section, the proofs that we have seen have been pretty easy. We end the section with a theorem about another of the triangle behaviors mentioned in Section 7.1.3. The proof of this theorem is very long and is not easy. You will study it in a homework exercise.

Theorem 58 the *SSS* congruence theorem for Neutral Geometry
In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 58

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that ΔABC and ΔDEF are given such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$.

Introduce the triangle ΔABP.

(2) Line $\overleftrightarrow{AB}$ creates two half planes. Let H_C be the half plane containing C, and let H_2 be the other one.

(3) There exists a ray $\overrightarrow{AG}$ such that point G is in half-plane H_2 and such that $\angle BAG \cong \angle EDF$. **(Justify.)**

(4) There exists a point P on ray $\overrightarrow{AG}$ such that $\overline{AP} \cong \overline{DF}$. **(Justify.)**

(5) $\Delta ABP \cong \Delta DEF$. **(Justify.)** Notice that this tells us that in addition to the congruences $\overline{AB} \cong \overline{DE}$ and $\angle BAG \cong \angle EDF$ and $\overline{AP} \cong \overline{DF}$ that we already knew about from statements (1), (3), (4), we now also have the congruences $\overline{BP} \cong \overline{EF}$ and $\angle ABP \cong \angle DEF$ and $\angle BPA \cong \angle EFD$.

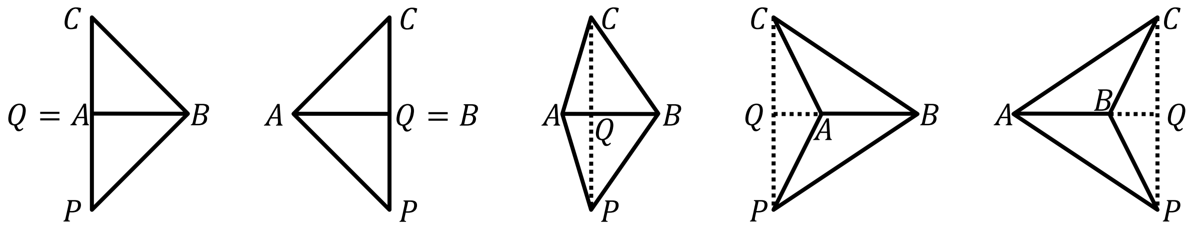
Introduce the point Q and five possibilities for it.

(6) Line $\overleftrightarrow{AB}$ must intersect segment $\overline{CP}$ at a point Q between C and P. **(Justify.)**

(7) There are five possibilities for where point Q can be on line $\overleftrightarrow{AB}$.

(i) $Q = A$.
(ii) $Q = B$.
(iii) $A * Q * B$.
(iv) $Q * A * B$.
(v) $A * B * Q$.

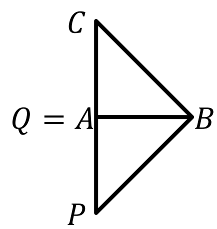
These possibilities are illustrated in the drawings below.



(i) $Q = A$ (ii) $Q = B$ (iii) $A * Q * B$ (iv) $Q * A * B$ (v) $A * B * Q$

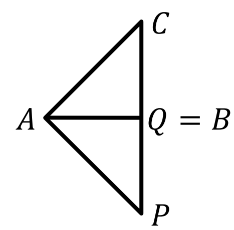
Case (i) $\boldsymbol{Q = A}$.

(8) Suppose $Q = A$.
(9) Then $\angle BCP \cong \angle BPC$. **(Justify.)**
(10) $\Delta ABC \cong \Delta ABP$. **(Justify.)**
(11) $\Delta ABC \cong \Delta DEF$. **(Justify.)**



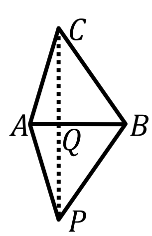
Case (ii) $\boldsymbol{Q = B}$.

(12) – (15) This case works just like Case (i), with the roles of A and B interchanged. You will write the details of the steps, and the justifications, in a homework exercise.



Case (iii) $\boldsymbol{A * Q * B}$.

(16) Suppose $A * Q * B$.
(17) $\angle ACP \cong \angle APC$. **(Justify.)**
(18) $\angle BCP \cong \angle BPC$. **(Justify.)**



Establish that point $\boldsymbol{Q}$ is in the interiors of two angles, then use congruent angle addition

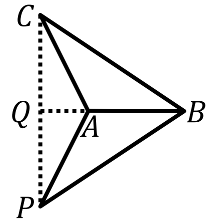
(19) Point Q is in the interior of $\angle ACB$. **(Justify.)**
(20) Point Q is in the interior of $\angle APB$. **(Justify.)**
(21) $\angle ACB \cong \angle APB$. **(Justify.)**
(22) $\Delta ABC \cong \Delta ABP$. **(Justify.)**
(23) $\Delta ABC \cong \Delta DEF$. **(Justify.)**

Case (iv) $\boldsymbol{Q * A * B}$.

(24) Suppose $Q * A * B$.

(25) $\angle ACP \cong \angle APC$. **(Justify.)**

(26) $\angle BCP \cong \angle BPC$. **(Justify.)**



Establish that point $\boldsymbol{A}$ is in the interiors of two angles, then use congruent angle subtraction

(27) Point A is in the interior of $\angle BCQ$. **(Justify.)**

(28) Point A is in the interior of $\angle BPQ$. **(Justify.)**

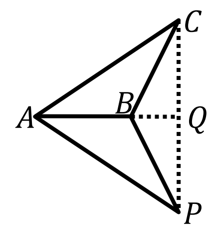
(29) $\angle ACB \cong \angle APB$. **(Justify.)**

(30) $\Delta ABC \cong \Delta ABP$. **(Justify.)**

(31) $\Delta ABC \cong \Delta DEF$. **(Justify.)**

Case (v) $\boldsymbol{A * B * Q}$.

(32) – (39) This case works just like Case (iv), with the roles of A and B interchanged.



Conclusion of cases.

(40) We see that $\Delta ABC \cong \Delta DEF$ in every case.

End of proof

The proof just finished is the longest in the book so far. The good news is that it is one of the longest in the entire book.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 7.8 on page 189.

7.3. Theorems about Bigger and Smaller Parts of Triangles

The triangle theorems that we studied in the previous section dealt with congruences—congruence of segments, or of angles, or of triangles. We now turn our attention to theorems involving segments or angles that are *not* congruent. The first of these theorems, called the Neutral Exterior Angle Theorem, has a difficult proof. We will need some terminology.

Definition 57 *exterior angle*, *remote interior angle*

An *exterior angle* of a triangle is an angle that forms a linear pair with one of the angles of the triangle. Each of the two other angles of the triangle is called a *remote interior angle* for that exterior angle. For example, a triangle ΔABC has six exterior angles. One of these is $\angle CBD$, where D is a point such that $A * B * D$. For the exterior angle $\angle CBD$, the two remote interior angles are $\angle ACB$ and $\angle CAB$.

Here is the Neutral Exterior Angle Theorem. You will justify the proof in a class drill

Theorem 59 Neutral Exterior Angle Theorem

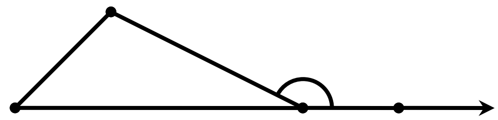
In Neutral Geometry, the measure of any exterior angle is greater than the measure of either of its remote interior angles.

Remark: The statement of the theorem can be illustrated by the picture below.

Theorem 59

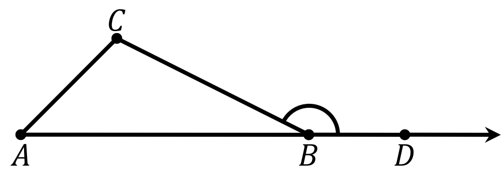
Proof

(1) Suppose that a triangle and an exterior angle are given.



Part I: Show that the measure of the given exterior angle is larger than the measure of the remote interior angle that the exterior point does not lie on.

(2) Label the points so that the triangle is ΔABC and the exterior angle is $\angle CBD$. The two remote interior angles are $\angle BAC$ and $\angle BCA$. Observe that point D lies on side $\overrightarrow{AB}$ of $\angle BAC$



but point D does not lie on either of the sides of angle $\angle BCA$. Our goal in Part I of the proof will be to show that $m(\angle CBD) > m(\angle BCA)$.

(3) There exists a point E that is the midpoint of side $\overline{BC}$. **(Justify.) (Make a drawing.)**

(4) $\overline{EB} \cong \overline{EC}$. **(Justify.) (Update your drawing.)**

(5) There exists a point F such that $A * E * F$. **(Justify.) (Make a new drawing.)**

(6) There exists a point G on ray $\overrightarrow{EF}$ such that $\overline{EG} \cong \overline{EA}$. **(Justify.) (Update your drawing.)**

(7) $\angle AEC \cong \angle GEB$. **(Justify.) (Make a new drawing.)**

(8) $\Delta AEC \cong \Delta GEB$. **(Justify.) (Make a new drawing.)**

Make observations about angles

(9) $\angle ACE \cong \angle GBE$. **(Justify.) (Make a new drawing.)**

Prove that Point G is in the interior of $\angle CBD$

(10) Points E and G are on the same side of line $\overleftrightarrow{BD}$. **(Justify.) (Make a new drawing.)**

(11) Points E and C are on the same side of line $\overleftrightarrow{BD}$. **(Justify.) (Make a new drawing.)**

(12) Therefore, points C and G are on the same side of line $\overleftrightarrow{BD}$. **(Justify.) (Make a new drawing.)**

(13) Points A and G are on opposite sides of line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(14) Points A and D are on opposite sides of line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(15) Therefore, points G and D are on the same side of line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(16) Conclude that point G is in the interior of $\angle CBD$. **(Justify.) (Make a new drawing.)**

Make some more observations about angles

(17) $m(\angle CBD) > m(\angle CBG)$. **(Justify.)**

(18) $m(\angle CBD) > m(\angle BCA)$. **(Justify.)**

Part II: Show that the measure of the given exterior angle is also larger than the measure of the remote interior angle that the exterior point does lie on.

(19) There exists a point H such that $C * B * H$. **(Justify.) (Make a new drawing.)** Observe that $\angle ABH$ is an exterior angle for ΔABC and also observe that the point H does not lie on the remote interior angle $\angle BAC$.
(20) $m(\angle ABH) > m(\angle BAC)$ (by statements identical to statements (2) through (12), but with points A, C, D replaced in all statements with points C, A, H.)
(21) $m(\angle ABH) = m(CBD)$. **(Justify.) (Make a new drawing.)**
(22) $m(\angle CBD) > m(\angle BAC)$. **(Justify.)**
End of proof

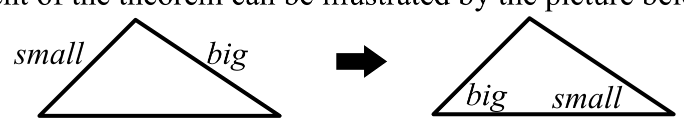
The following corollary is a very simple application of the exterior angle theorem. You will be asked to prove it in a homework exercise.

Theorem 60 (Corollary) If a triangle has a right angle, then the other two angles are acute.

In the previous section, we studied the *CS* ➔ *CA* theorem and the *CA* ➔ *CS* theorem, both having to do with congruent sides and congruent angles in a triangle. (These were combined into the *CACS* theorem.) The following three theorems are analogous, but have to do with sides and angles that are *not* the same size. You will justify the steps in the proof of the first of the three in a homework exercise.

Theorem 61 the *BS* ➔ *BA* theorem for triangles in Neutral Geometry
In Neutral Geometry, if one side of a triangle is longer than another side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side. That is, in a triangle, if *BS* then *BA*.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 61

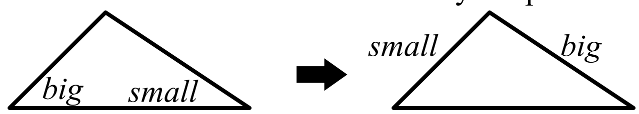
Proof
(1) Suppose that ΔABC has $AB > AC$. **(Make a drawing.)**
(2) There exists a point D on ray $\overrightarrow{AB}$ such that $\overline{AD} \cong \overline{AC}$. **(Justify.) (Make a new drawing.)**
(3) $A * D * B$. **(Justify.)**
(4) Point D is in the interior of $\angle ACB$. **(Justify.) (Make a new drawing.)**
(5) $m(\angle ACB) > m(\angle ACD)$. **(Justify.) (Make a new drawing.)**
(6) $m(\angle ACD) = m(\angle ADC)$. **(Justify.) (Make a new drawing.)**
(7) $m(\angle ADC) > m(\angle ABC)$. **(Justify.) (Make a new drawing.)**
(8) $m(\angle ACB) > m(\angle ABC)$. **(Justify.) (Make a new drawing.)**
End of proof

The next theorem has a fun proof in which we are able to avoid making a complicated drawing by being clever about applying earlier theorems. I have supplied justifications for all steps.

Theorem 62 the *BA* ➔ *BS* theorem for triangles in Neutral Geometry

In Neutral geometry, if the measure of one angle is greater than the measure of another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle. That is, in a triangle, if *BA* then *BS*.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 62

Proof (Method: Prove the contrapositive statement ~*BS* ➔ ~*BA*.)

(1) Suppose that in ΔABC, the inequality $AB > AC$ is false.
(2) There are two possibilities: either (i) $AB = AC$ or (ii) $AC > AB$.

Case (i) $\boldsymbol{AB = AC}$

(3) Suppose that $AB = AC$. (assumption) **(Make a drawing.)**
(4) Then $m(\angle ACB) = m(\angle ABC)$. (by *CS*➔*CA* theorem applied to ΔABC) So in this case, the inequality $m(\angle ACB) > m(\angle ABC)$ is false.

Case (ii) $\boldsymbol{AC} > AB$

(5) Suppose that $AC > AB$. (assumption) **(Make a new drawing.)**
(6) Then $m(\angle ABC) > m(\angle ACB)$. (by *BS*➔*BA* theorem applied to ΔABC) So in this case, the inequality $m(\angle ACB) > m(\angle ABC)$ is also false.

Conclusion of cases

(7) We see that in both cases, the inequality $m(\angle ACB) > m(\angle ABC)$ is also false.

End of proof

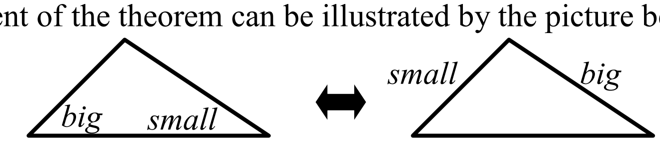
Consider what we know about bigger sides and bigger angles in triangles.

- From Theorem 61, we know that *BS* ➔ *BA*.
- From Theorem 62, we know that *BA* ➔ *BS*.

Combining, we have the following immediate corollary:

Theorem 63 (Corollary) The *BABS* theorem for triangles in Neutral Geometry.
In any triangle in Neutral Geometry, bigger angles are always opposite bigger sides. That is, *BA* ⇔ *BS*.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 63

So far in this section, we studied theorems that compare the sizes of angles or segments. Our final theorem of the section is the Triangle Inequality for Neutral Geometry. It involves the sum of the lengths of segments. You will justify the steps of the proof in a homework exercise.

Theorem 64 The Triangle Inequality for Neutral Geometry
In Neutral Geometry, the length of any side of any triangle is less than the sum of the lengths of the other two sides.
That is, for all non-collinear points A, B, C, the inequality $AC < AB + BC$ is true.

Proof

(1) Suppose that a triangle is given, and a side has been chosen on that triangle. (The goal is to show that the length of the chosen side is less than the sum of the lengths of the other two sides.) **(Make a drawing.)**
(2) Label the triangle ΔABC with $\overline{AC}$ the chosen side. (Now the goal is to show that $AC < AB + BC$.) **(Update your drawing.)**
(3) There exists a point D such that $A * B * D$. **(Justify.) (Make a new drawing.)**
(4) There exists a point E on ray $\overrightarrow{BD}$ such that $\overline{BE} \cong \overline{BC}$. **(Justify.) (Make a new drawing.)**
(5) $AB + BC = AB + BE = AE$. **(Justify.)**
(6) Point B is in the interior of angle $\angle ACE$. **(Justify.) (Make a new drawing.)**
(7) $m(\angle BEC) = m(\angle BCE)$. **(Justify.) (Make a new drawing.)**
(8) $m(\angle BCE) < m(\angle ACE)$. **(Justify.) (Make a new drawing.)**
(9) $m(\angle BEC) < m(\angle ACE)$. **(Justify.) (Make a new drawing.)**
(10) $m(\angle AEC) < m(\angle ACE)$. **(Justify.) (Make a new drawing.)**
(11) $AC < AE$. **(Justify.) (Make a new drawing.)**
(12) $AC < AB + BC$. **(Justify.)**

End of proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 7.8 on page 189.

7.4. Advanced Topic: Properties of the Distance Function

In this section, we will examine properties of the Distance Function. Because the distance function is the subject of our investigation, we will not hide it in the notation. That is, we will write $d(A, C)$ instead of AC, for example. With that more explicit notation, the "triangle inequality" just proven in Theorem 64 reads as follows.

For all non-collinear points A, B, C, the inequality $d(A, C) < d(A, B) + d(B, C)$ is true.

There are two important subtleties in this statement.

- Points A, B, C must be noncollinear (because they are vertices of a triangle).
- The inequality is strict. That is, the symbol is $<$ rather than $\leq$.

It is worthwhile to consider what can be said about the three distances if we relax the restriction on the three points. That is, if we allow the letters P, Q, R to stand for points that are not necessarily collinear and not even necessarily distinct, then what can be said about the three distances $d(P, R)$ and $d(P, Q)$ and $d(Q, R)$?

The answer to that question is the following statement.

For all points P, Q, R, the inequality $d(P, R) \leq d(P, Q) + d(Q, R)$ is true.

The two important subtleties in this statement are

- The symbols P, Q, R represent *any* three points, not necessarily collinear or even distinct.
- The inequality is inclusive. That is, the symbol is $\leq$ rather than $<$.

This new statement is commonly called the *triangle inequality*. That name is bad for two reasons:

- The three letters P, Q, R do not necessarily represent the vertices of a triangle
- We have already seen a different statement called the triangle inequality, in Theorem 64.

We will distinguish our new statement from our previous statement by calling the new one the *Distance Function Triangle Inequality*. Its restatement and proof follow. The bad news about the proof is that there are seven cases to consider. The good news is that in each case, previous results tell us that the claim is true. So the proof gives us a nice opportunity to review previous results and to see them put together to prove something new without any new work.

Theorem 65 The Distance Function Triangle Inequality for Neutral Geometry
The function d satisfies the *Distance Function Triangle Inequality*.
That is, for all points P, Q, R, the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true.

Proof

(1) Let P, Q, R be any three points, not necessarily collinear or even distinct.
(2) There are seven possibilities:
(i) All three letters refer to the same point. That is, $P = Q = R$.
(ii) $P = Q$ but R is distinct.
(iii) $Q = R$ but P is distinct.
(iv) $R = P$ but Q is distinct.
(v) The three points are distinct and collinear and $P * Q * R$ is true.
(vi) The three points are distinct and collinear and $P * Q * R$ is not true.
(vii) The three points are non-collinear.

Case (i)

(3) If $P = Q = R$, then $d(P,R) = 0$ and $d(P,Q) = 0$ and $d(Q,R) = 0$. Substituting these values into the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$, it becomes $0 \leq 0 + 0$. This inequality is true. So the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true in this case.

Case (ii)

(4) If $P = Q$ but R is distinct, then $d(P,Q) = 0$ and $d(Q,R) = d(P,R)$. Substituting these into the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$, it becomes $d(P,R) \leq 0 + d(P,R)$. This inequality is true. So the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true in this case.

Case (iii)

(5) If $Q = R$ but P is distinct, then $d(Q,R) = 0$ and $d(P,Q) = d(P,R)$. Substituting these into the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$, it becomes $d(P,R) \leq d(P,R) + 0$. This inequality is true. So the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true in this case.

Case (iv)

(6) If $R = P$ but Q is distinct, then $d(P,R) = 0$ and $d(P,Q) = d(Q,R)$. Substituting these into the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$, it becomes $0 \leq d(P,Q) + d(P,Q)$. This inequality is true. So the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true in this case.

Case (v)

(7) If the three points are distinct and collinear and $P * Q * R$ is true, then Theorem 16 in Section 4.1.2 tells us that the equation $d(P,R) = d(P,Q) + d(Q,R)$ is true. So the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is also true.

Case (vi)

(8) If the three points are distinct and collinear and $P * Q * R$ is not true, then Theorem 17 in Section 4.1.3 tells us that the strict inequality $d(P,R) < d(P,Q) + d(Q,R)$ is true. So the weaker inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is also true.

Case (vii)

(9) If the three points are noncollinear, then Theorem 64 from the previous section tells us that the strict inequality $d(P,R) < d(P,Q) + d(Q,R)$ is true. So the weaker inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is also true.

Conclusion of cases

(10) We see that in every case, the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true.

End of proof

Since Theorem 64 and Theorem 65 are so similar, it is worth putting them next to each other and discussing the contexts in which each is useful.

Theorem 64 (The Triangle Inequality) says

For all non-collinear points A, B, C, the inequality $d(A,C) < d(A,B) + d(B,C)$ is true.

Theorem 65 (The Distance Function Triangle Inequality) says

For all points P, Q, R, the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true.

Theorem 64 (The Triangle Inequality) will be used in settings where one is dealing with a triangle. That is, if is known that three points *are* non-collinear, then it makes sense to make use of the full strength of the strict inequality. In this book, we will always be using Theorem 64, and we will use it to make statements about triangles.

Theorem 65 (The Distance Function Triangle Inequality) is relevant to discussions about properties of distance functions. I will explain.

In higher mathematics, the expression *distance function* is used in a very precise way. Given a set S, the expression *"a distance function on S"* means a function $d: S \times S \to \mathbb{R}$ that has the following three properties

(1) d is *positive definite*: For all elements P and Q of set S, $d(P,Q) \geq 0$, and $d(P,Q) = 0$ if and only if $P = Q$. That is, if and only if P and Q are actually the same element of set S.
(2) d is *symmetric*: For all elements P and Q of set S, $d(P,Q) = d(Q,P)$.
(3) d satisfies the *distance function triangle inequality*: For all elements P, Q, R of set S, the inequality $d(P,R) \leq d(P,Q) + d(Q,R)$ is true.

Here we should observe an important subtlety and imprecision in our axiom system. Notice that axiom <N4> reads

> <N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

That axiom calls the function d the *Distance Function on the Set of Points*, but the axiom does not actually say that d has the properties normally associated with the term *distance function*. Later on, it is proven in theorems that the function d does have those properties.

- Theorem 8 proves that the function d is Positive Definite.
- Theorem 9 proves that the function d is Symmetric.
- Theorem 65 proves that d satisfies the Distance Function Triangle Inequality.

So it is imprecise writing for axiom <N4> (The Distance Axiom) to call the function d the *Distance Function on the Set of Points* before it has been proven that d does in fact have the properties of a distance function. Perhaps a more precise, more correct presentation would be as follows

- Axiom <N4> would say that a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ exists, but it would not call d the distance function on the set of points.
- Theorem 8 proves that the function d is Positive Definite.
- Theorem 9 proves that the function d is Symmetric.
- Theorem 65 proves that d satisfies the Distance Function Triangle Inequality.
- Then an observation would be made that the function d is qualified to be called a Distance Function for the Set of Points.

But it is common in math books for functions to be introduced and given names that seem to imply that the function has certain properties *before* it has been proven that the function has those properties. This can be confusing. It is also tricky, because the writer must be careful to not assume any property of the function before the property has been proven. For example, in this book, the function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ is introduced in Axiom <N4> and is called the Distance Function on the Set of Points. This name seems to imply that the function d is symmetric, but in fact the symmetry property is not guaranteed by the axiom. And this book does not assume that the function d is symmetric. The symmetry property is proven in Theorem 9. After that point in the book, the fact can be used that d is symmetric. But when the symmetry property is used later in the book, it is always justified by Theorem 9. It is not justified by the fact that axiom <N4> happens to call d the *distance function function on the set of points*.

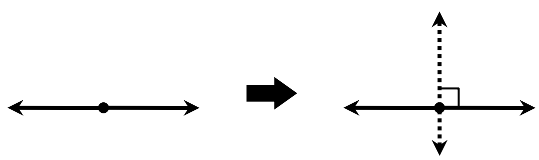
That is the end of our discussion of properties of the Distance Function. We will resume using the abbreviated notation for the distance between two points. For example, we will write AC instead of $d(A, C)$.

7.5. More About Perpendicular Lines

In the previous chapter, we defined the concept of *perpendicular* and studied a number of theorems involving the concept. The only axioms that were used in that study were the Axioms of Incidence and Distance <N1> through <N5>, the Axiom of Separation <N6>, and the Axioms of Angle Measurement <N7> through <N9>. The chapter made no use of the Axiom of Triangle Congruence <N10>. In the current chapter, we *are* using axiom <N10>. That axiom does not mention perpendicular lines, but it turns out that the axiom does indirectly give us new information about the behavior of perpendicular lines. In this section, we will study perpendicular lines, making use axiom <N10> and the theorems that follow from it.

7.5.1. The Perpendicular from a Point to a Line

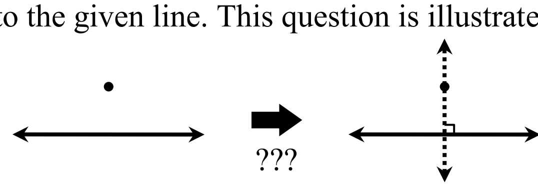
In Section 6.6, we were able to to prove Theorem 46 about the existence and uniqueness of a line that is perpendicular to a given line and that passes through *a given point that* ***lies on*** *the given line*.



Theorem 46

The proof makes use of earlier theorems, and those theorems rely only on axioms <N2> - <N8>.

In Section 6.6, we also considered the different situation where there is a given line and *a given point that* ***does not lie on*** *the given line*. It was pointed out that Theorem 46 does not apply to this situation. It was natural to wonder if there exists a unique line that passes through the given point and is perpendicular to the given line. This question is illustrated by the drawing below.



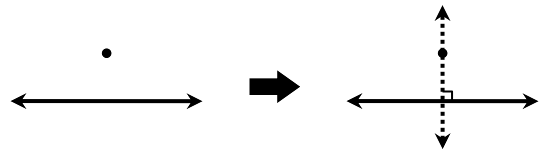
It turns out that there does exist a unique line that passes through the given point and is perpendicular to the given line, but the proof requires theorems that follow from axiom <N10>.

Now that we have introduced that axiom and studied some theorems that follow from it, we are ready to prove the existence and uniqueness of the perpendicular from a point to a line. Part 1 of the proof –the proof of existence—will use theorems of angle congruence and triangle congruence. Part 2 of the proof—the proof of uniqueness—will use Theorem 60, a theorem whose proof depended on the Exterior Angle Theorem for Neutral Geometry.

Theorem 66 existence and uniqueness of a line that is perpendicular to a given line through a given point that *does not lie on* the given line

For any given line and any given point that *does not lie on* the given line, there is exactly one line that passes through the given point and is perpendicular to the given line.

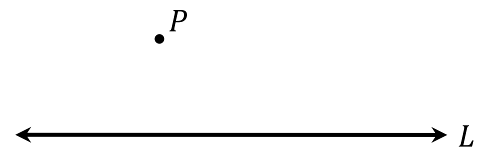
Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 66

Proof

(1) Suppose that L is a line and P is a point that does not lie on L.



Part 1: Show that a line M exists that passes through P and is perpendicular to L.

(2) There exist distinct points Q, R, S on L. **(Justify.)**

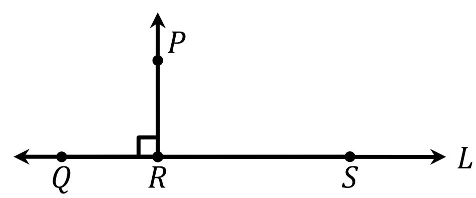
(3) There are two possibilities for angle $\angle PRQ$.

(i) Angle $\angle PRQ$ is a right angle.

(ii) Angle $\angle PRQ$ is not a right angle.

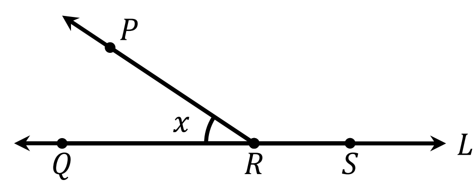
Case (i) Angle $\angle PRQ$ is a right angle.

(4) Suppose that angle $\angle PRQ$ is a right angle. In this case, line $\overleftrightarrow{PQ}$ is perpendicular to line L. We can let M be line $\overleftrightarrow{PR}$.



Case (ii): Angle $\angle PRQ$ is not a right angle.

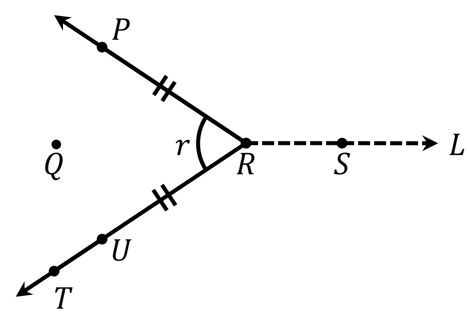
(5) Suppose that angle $\angle PRQ$ is a not right angle. Without loss of generality, we may assume that angle $\angle PRQ$ is acute. (If $\angle PRQ$ is not a right angle and is not acute, then it will be obtuse. But in that case, the angle $\angle PRS$ will be acute. In that case, we can interchange the names of points Q and S so that the angle $\angle PRQ$ will be acute.)Let $x = m(\angle PRQ)$, so that $0 < x < 90$.



(6) Line $\overleftrightarrow{RP}$ creates two half planes. Let H_Q be the half plane containing Q. Let $r = 2x$. Note that $0 < r < 180$.

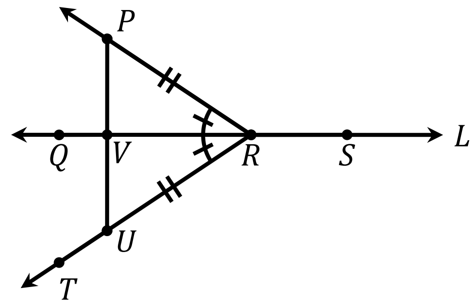
(7) There exists a ray $\overrightarrow{RT}$ such that point T is in half-plane H_Q and $m(\angle PRT) = r = 2x$. **(Justify.)**

(8) There exists a point U on ray $\overrightarrow{RT}$ such that $\overline{RU} \cong \overline{RP}$. **(Justify.)**



(9) Point Q is in the interior of angle $\angle PRU$. (By Theorem 39 (II) ➔ (I) applied to points Q and U on the same side of line $\overleftrightarrow{RP}$)

(10) Ray $\overrightarrow{RQ}$ intersects segment $\overline{PU}$. **(Justify.)** Let V be the point of intersection.

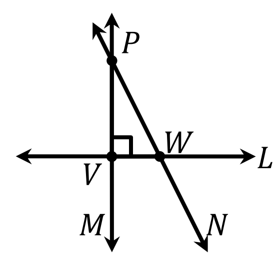


(11) $\angle VRU \cong \angle VRP$. (by (5,6,7) and angle addition axiom)
(12) $\Delta VRU \cong \Delta VRP$. **(Justify.)**
(13) $\angle RVU \cong \angle RVP$. **(Justify.)**
(14) Angles $\angle RVU$ and $\angle RVP$ form a linear pair. **(Justify.)**

(15) $m(\angle RVP) = 90$. **(Justify.)** So line $\overleftrightarrow{PU}$ is perpendicular to line L. We can let M be line $\overleftrightarrow{PU}$.

Conclusion of cases.
(16) We see that in every case, there exists a line M that passes through P and is perpendicular to line L.
End of proof Part 1.

Proof Part 2: Show that no other lines that pass through *P* are perpendicular to *L*.



(17) Suppose that a line N passes through P and intersects line L at a point W distinct from V.
(18) Angle $\angle PWV$ is acute. (by Theorem 60 applied to triangle ΔPVW with right angle $\angle PVW$.) So line N is not perpendicular to L.
End of Proof

Now that we have proven the existence and uniqueness of the line perpendicular to a given line through a given point not on the given line, we can refer to "*the* perpendicular from a point to a line". The following theorem is about the perpendicular from a point to line. The theorem has a very easy proof. You will justify the proof steps in a homework exercise.

Theorem 67 The shortest segment connecting a point to a line is the perpendicular.
Proof
(1) Suppose that L is a line and P is a point not on L, and that Q, R are two points on L such that line $\overleftrightarrow{PQ}$ is perpendicular to L.
(2) $\angle PRQ$ is an acute angle. **(Justify.)**
(3) $PQ < PR$. **(Justify.)**
End of proof

7.5.2. Altitudes in Neutral Geometry

The previoius section was about perpendicular lines. In this section, we will study study right triangles and triangle altitude lines, both of which involve perpendicular lines. Here is our first definition:

Definition 58 right triangle, and hypotenuse and legs of a right triangle
A *right triangle* is one in which one of the angles is a right angle. Recall that Theorem 60 states that if a triangle has one right angle, then the other two angles are acute, so there can

only be one right angle in a right triangle. In a right triangle, the side opposite the right angle is called the *hypotenuse* of the triangle. Each of the other two sides is called a *leg* of the triangle.

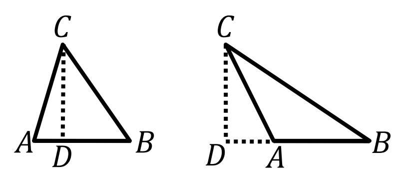
Our first theorem of the section has such an easy proof that the theorem could really be considered a corollary. You will prove the theorem in an exercise

Theorem 68 In any right triangle in Neutral Geometry, the hypotenuse is the longest side.

Our second theorem of the section will involve the concept of an *altitude line*, the *foot of an altitude*, and an *altitude segment* of a triangle. Here are definitions.

Definition 59 altitude line, foot of an altitude line, altitude segment
An *altitude line* of a triangle is a line that passes through a vertex of the triangle and is perpendicular to the opposite side. (Note that the altitude line does not necessarily have to intersect the opposite side to be perpendicular to it. Also note that Theorem 66 in the previous subsection tells us that there is exactly one altitude line for each vertex.) The point of intersection of the altitude line and the line determined by the opposite side is called the *foot of the altitude line*. An *altitude segment* has one endpoint at the vertex and the other endpoint at the foot of the altitude line drawn from that vertex. For example, in triangle ΔABC, an *altitude line* from vertex A is a line L that passes through A and is perpendicular to line $\overleftrightarrow{BC}$. The *foot* of altitude line L is the point D that is the intersection of line L and line $\overleftrightarrow{BC}$. The *altitude segment* from vertex A is the segment $\overline{AD}$. Point D can also be called the foot of the altitude segment $\overline{AD}$.

It is worth noting that an altitude segment does not always stay inside the triangle. Consider, for example, the two drawings at right. In the first drawing, altitude segment $\overline{CD}$ is inside the triangle. In the second drawing it is not.



As with so many simple observations about behavior of objects in drawings, it can be very tedious to articulate and prove abstract statements about the corresponding behavior in Neutral Geometry. The following theorem makes a very limited claim, and yet the proof is fairly tricky.

Theorem 69 (Lemma) In any triangle in Neutral Geometry, the altitude to a longest side intersects the longest side at a point between the endpoints.
Given: Neutral Geometry triangle ΔABC, with point D the foot of the altitude line drawn from vertex C to line $\overleftrightarrow{AB}$.
Claim: If $\overline{AB}$ is a longest side (that is, if $AB \geq CB$ and $AB \geq CA$), then $A * D * B$.

Proof (for readers interested in advanced topics and for graduate students)
(Prove the contrapositive)

(1) Suppose that in Neutral Geometry triangle ΔABC is given, and that point D is the foot of the altitude line drawn from vertex C to line $\overleftrightarrow{AB}$, and that $A * D * B$ is not true.

(2) Then there are four possibilities for where point D can be on line $\overleftrightarrow{AB}$.

(i) $D = A$.

(ii) $D = B$.
(iii) $D * A * B$.
(iv) $A * B * D$.

Case (i) $D = A$.

(3) Suppose $D = A$. **(Make a drawing.)** Then $\angle CAB$ is a right angle, and so ΔABC is a right triangle with hypotenuse $\overline{BC}$.
(4) $BC > AB$ **(Justify.)**
(5) We see that in this case, side $\overline{AB}$ is not a longest side.

Case (ii) $D = B$.

(6) – (8) Suppose $D = B$. **(Make a new drawing. Fill in the proof steps to show that in this case, side $\overline{AB}$ is not a longest side.)**

Case (iii) $D * A * B$.

(9) Suppose $D * A * B$. **(Make a new drawing.)** Then ΔDBC is a right triangle with hypotenuse $\overline{BC}$.
(10) $BC > DB$ **(Justify.)**
(11) $DB > AB$ **(Justify.)**
(12) $BC > AB$ **(Justify.)**
(13) We see that in this case, side $\overline{AB}$ is not a longest side.

Case (iv) $A * B * D$.

(14) – (18) Suppose $A * B * D$. **(Make a new drawing. Fill in the proof steps to show that in this case, side $\overline{AB}$ is not a longest side.)**

Conclusion of Cases

(19) We see that in every case, side $\overline{AB}$ is not a longest side.

End of proof

As was mentioned before the theorem, the claim of the theorem is fairly limited. By that, I mean that the theorem does not address many of the configurations of altitudes in triangles that can occur. To get an idea of what sorts of configurations the theorem does *not* address, notice that the theorem statement involves a conditional statement:

Claim: If $\overline{AB}$ is the longest side (that is, if $AB > BC$ and $AB > BC$), then $A * D * B$.

The theorem says nothing about what may happen if $\overline{AB}$ is *not* the longest side. Indeed, if $\overline{AB}$ is not the longest side, a variety of things can happen.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 7.8 on page 189.

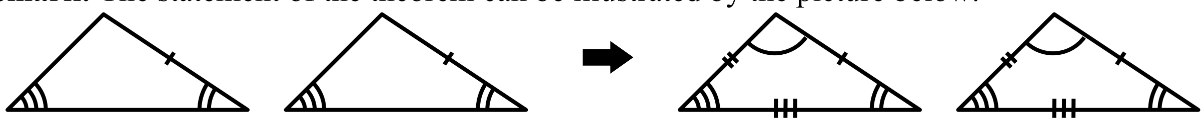
7.6. A Final Look at Triangle Congruence in Neutral Geometry

In Section 7.1.3, we discussed four behaviors of drawn triangles that we hoped would be behaviors of abstract triangles in Neutral Geometry as well. The first behavior that we described for drawings was guaranteed to also be manifest in Neutral Geometry triangles because Neutral Axiom <N10> (the *SAS* Congruence Axiom) guaranteed it. The second and third behaviors that we described for drawings were found to be manifest in Neutral Geometry as well, but it was not because they were guaranteed explicitly by axioms. Rather, we proved in Theorem 54 (the *ASA*

Congruence Theorem) and Theorem 58 (the *SSS* Congruence Theorem) that those behaviors would be manifest in Neutral Geometry. Here, we return to the fourth behavior of drawn triangles from that discussion in Section 7.1.3. We will prove that the fourth behavior will also be manifest in Neutral Geometry triangles. The theorem is the *AAS* Congruence Theorem; you will justify its proof in a class drill

Theorem 70 the Angle-Angle-Side (*AAS*) Congruence Theorem for Neutral Geometry
In Neutral Geometry, if there is a correspondence between parts of two right triangles such that two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 70

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that in Neutral Geometry, triangles ΔABC and ΔDEF have $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$.

(2) There exists a point G on ray $\overrightarrow{BA}$ such that $\overline{BG} \cong \overline{ED}$. **(Justify.)**

(3) $\Delta GBC \cong \Delta DEF$. **(Justify.)**

(4) $\angle CGB \cong \angle FDE$. **(Justify.)**

(5) $\angle CGB \cong \angle CAB$. **(Justify.)**

(6) There are three possibilities for where point G can be on ray $\overrightarrow{BA}$.

(i) $A * G * B$.

(ii) $G * A * B$.

(iii) $G = A$.

Case (i) $\boldsymbol{A * G * B}$.

(7) Suppose $A * G * B$. **(Make a drawing.)** Then $\angle CGB$ is an exterior angle for ΔCGA, and $\angle CAB$ is one of its remote interior angles.

(8) $m(\angle CGB) > m(\angle CAB)$ **(Justify.)**

(9) We have reached a contradiction. **(Explain the contradiction.)** Therefore, the assumption in step (7) was wrong.

Case (ii) $\boldsymbol{G * A * B}$.

(10) – (12) Suppose $G * A * B$. **(Make a new drawing. Fill in the proof steps to show that we reach a contradiction.)**

Conclusion of Cases

(13) Since Cases (i) and (ii) lead to contradictions, we conclude that only Case (iii) is possible. That is, it must be that $G = A$. Therefore, $\Delta ABC \cong \Delta DEF$.

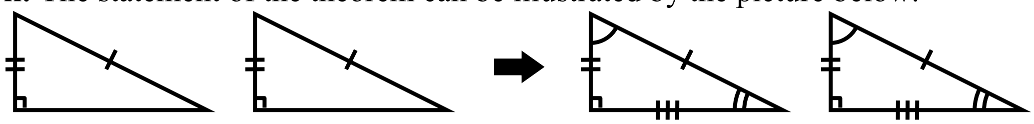
End of proof

Only one more congruence theorem left for Neutral Geometry! This last one is the Hypotenuse Leg Congruence Theorem for Neutral Geometry. Its proof uses the Angle-Angle-Side Congruence Theorem just proven.

Theorem 71 the Hypotenuse Leg Congruence Theorem for Neutral Geometry

> In Neutral Geometry, if there is a one-to-one correspondence between the vertices of any two right triangles, and the hypotenuse and a side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 71

Proof

(1) Suppose that in Neutral Geometry, right triangles ΔABC and ΔDEF have right angles at $\angle A$ and $\angle D$, and congruent hypotenuses $\overline{BC} \cong \overline{EF}$, and a congruent leg $\overline{AB} \cong \overline{DE}$, **(Make a drawing.)**

(2) There exists a point G such that $C * A * G$ and $\overline{AG} \cong \overline{DF}$. **(Justify. It will take two statements.) (Make a new drawing.)**

(3) $\Delta ABG \cong \Delta DEF$. **(Justify.) (Make a new drawing.)**

(4) $\overline{BG} \cong \overline{EF}$. **(Justify.) (Make a new drawing.)**

(5) $\overline{BG} \cong \overline{BC}$. **(Justify.) (Make a new drawing.)**

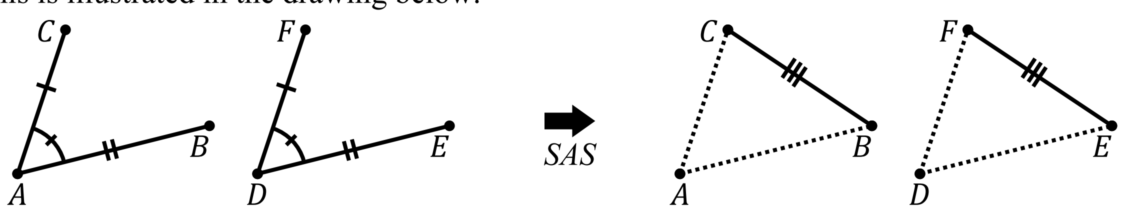
(6) $\angle CGB \cong \angle GCB$. **(Justify.) (Make a new drawing.)**

(7) $\Delta ABC \cong \Delta ABG$. (by Theorem 70, the *AAS* Congruence Theorem for Neutral Geometry)

(8) $\Delta ABC \cong \Delta DEF$. **(Justify.)**

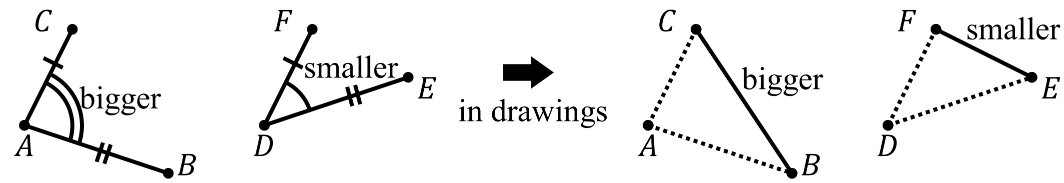
End of proof

We have one leftover theorem that deals with lengths of sides and measures of angles in triangles of Neutral Geometry. Consider the following situation: Suppose that ΔABC and ΔDEF have two pairs of corresponding sides that are congruent, $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. If the included angles are congruent, $\angle BAC \cong \angle EDF$, then Neutral Axiom <N10> (the *SAS* axiom) tells us that $\Delta ABC \cong \Delta DEF$. Therefore, the third sides of the triangles are also congruent. That is, $\overline{BC} \cong \overline{EF}$.This is illustrated in the drawing below.



(Note, it is the *SAS* Axiom, not the *CA*➔*CS* theorem, that gives us the information about the congruent third sides. The *CA*➔*CS* theorem (Theorem 55) cannot be used in this situation because the congruent angles are in different triangles!!)

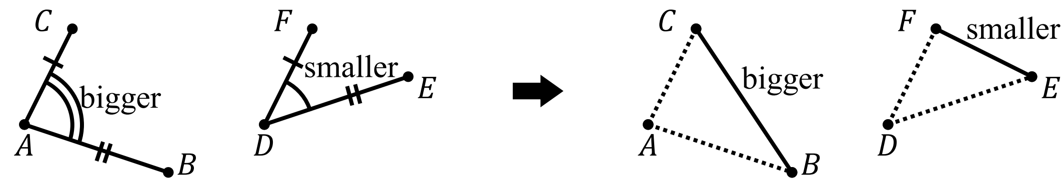
What if the included angles are *not* congruent? That is, suppose $m(\angle BAC) > m(\angle EDF)$. What can be said about the triangles in that case? We know what the answer would be in the same situation involving drawings: side $\overline{BC}$ of drawn triangle ΔABC would be longer than side $\overline{EF}$ of drawn triangle ΔDEF. This is illustrated in the drawing below.



Well, it turns out that abstract triangles in Neutral Geometry will exhibit this same behavior. The Hinge Theorem proves it.

Theorem 72 the Hinge Theorem for Neutral Geometry
In Neutral Geometry, if triangles ΔABC and ΔDEF have $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ and $m(\angle A) > m(\angle D)$, then $BC > EF$.

Remark: The statement of the theorem can be illustrated by the picture below.



Theorem 72

The proof of the Hinge Theorem is very long and involves a few cases. Studying the proof would not add significantly to our understanding, so I have not included a proof of the theorem in this book.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 7.8 on page 189.

7.7. Parallel lines in Neutral Geometry

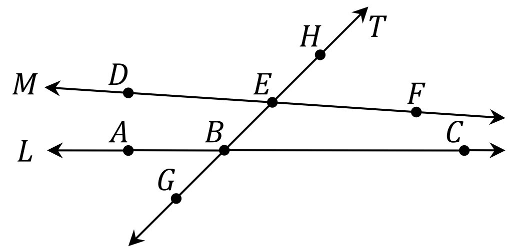
In this final section of the chapter, we will study an extremely important theorem called the Alternate Interior Angle Theorem for Neutral Geometry. In order to understand the wording of the Alternate Interior Angle Theorem, we need some definitions.

Definition 60 transversal
Words: Line T is transversal to lines L and M.
Meaning: Line T intersects L and M in distinct points.

Note that if line T is transversal to lines L and M, then L and M must be distinct lines. (The reason is that if L and M were the same line, then line T would not be able to intersect the lines in distinct points.) But lines L and M are not required to be parallel.

Definition 61 alternate interior angles, corresponding angles, interior angles on the same side of the transversal
Usage: Lines L, M, and transversal T are given.
Labeled Points:

Let B be the intersection of lines T and L, and let E be the intersection of lines T and M. (By definition of transversal, B and E are not the same point.) By Theorem 15, there exist points A and C on line L such that $A * B * C$, points D and F on line M such that $D * E * F$, and points G and H on line T such that $G * B * E$ and $B * E * H$. Without loss of generality, we may assume that points D and F are labeled such that it is point D that is on the same side of line T as point A. (See the figure at right.)



Meaning:

Special names are given to the following eight pairs of angles:

- The pair $\{\angle ABE, \angle FEB\}$ is a pair of alternate interior angles.
- The pair $\{\angle CBE, \angle DEB\}$ is a pair of alternate interior angles.
- The pair $\{\angle ABG, \angle DEG\}$ is a pair of corresponding angles.
- The pair $\{\angle ABH, \angle DEH\}$ is a pair of corresponding angles.
- The pair $\{\angle CBG, \angle FEG\}$ is a pair of corresponding angles.
- The pair $\{\angle CBH, \angle FEH\}$ is a pair of corresponding angles.
- The pair $\{\angle ABE, \angle DEB\}$ is a pair of interior angles on the same side of the transversal.
- The pair $\{\angle CBE, \angle FEB\}$ is a pair of interior angles on the same side of the transversal.

We will be interested in the situation where it is known that a pair of alternate interior angles is congruent. Which pair? Well, it turns out that if one pair of alternate interior angles is congruent, then the other pair is, as well. And each pair of corresponding angles is congruent. And each pair of interior angles on the same side of the transversal has measures that add up to 180. This is not difficult to show, and is articulated in the following theorem. You will prove the theorem in a homework exercise.

Theorem 73 Equivalent statements about angles formed by two lines and a transversal in Neutral Geometry

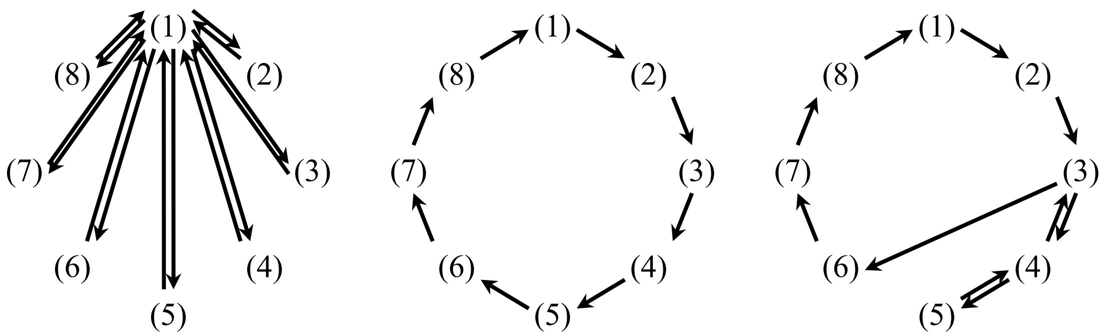
Given: Neutral Geometry, lines L and M and a transversal T, with points $A, \cdots, H$ labeled as in Definition 61, above.

Claim: The following statements are equivalent:

(1) The first pair of alternate interior angles is congruent. That is, $\angle ABE \cong \angle FEB$.
(2) The second pair of alternate interior angles is congruent. That is, $\angle CBE \cong \angle DEB$.
(3) The first pair of corresponding angles is congruent. That is, $\angle ABG \cong \angle DEG$.
(4) The second pair of corresponding angles is congruent. That is, $\angle ABH \cong \angle DEH$.
(5) The third pair of corresponding angles is congruent. That is, $\angle CBG \cong \angle FEG$.
(6) The fourth pair of corresponding angles is congruent. That is, $\angle CBH \cong \angle FEH$.
(7) The first pair of interior angles on the same side of the transversal has measures that add up to 180. That is, $m(\angle ABE) + m(\angle DEB) = 180$.
(8) The second pair of interior angles on the same side of the transversal has measures that add up to 180. That is, $m(\angle CBE) + m(\angle FEB) = 180$.

A Remark about Proving an Equivalence Theorem

There are many strategies possible for proving the theorem above. For example, here are three.



Strategy 1 Strategy 2 Strategy 3

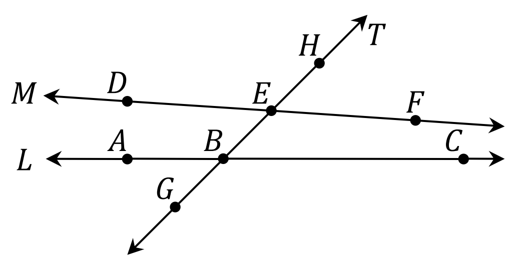
Strategy 1 is very simple conceptually, but notice that it involves fourteen proofs. Strategy 2 is also very simple conceptually, and it involves only eight proofs. But it might turn out that some of the proofs are difficult. (Maybe it is very hard to prove that (5) ➔ (6).) It might turn out to be easiest to build a proof using a strategy that is kind of a mixture of Strategy 1 and Strategy 2. One such proof is shown above as Strategy 3. You will prove the theorem in an exercise, and the choice of strategy will be up to you.

A Remark about "Using" an Equivalence Theorem

It is very important to keep in mind that Theorem 73 does not say that any of the eight statements are true. It only says that they are either all true or they are all false. This means that in any situation where one wants to use Theorem 73 as a justification for a step in a proof, it can only come after a prior step in which one of the statements (1) - (8) has already been proven true or false by some other means. That sounds vague, so I will give an example.

Consider the following fragment of the proof of the Wedgie Theorem. Its steps are lettered, rather than numbered.

Proof of the Wedgie Theorem
(*) some statements here
(q) $\angle ABH \cong \angle DEH$
(r) $\angle CBE \cong \angle DEB$
(*) some more statements here
End of proof



Observe that statements (q) and (r) in the proof of the Wedgie Theorem are just statements (4) and (2) in the list from Theorem 73. Assume that none of the statements (1) - (8) from Theorem 73 have shown up anywhere in the earlier steps (a) - (p) of the proof of the Wedgie Theorem. The question is, how can we justify statements (q) and (r) in the proof of the Wedgie Theorem?

Realize that Theorem 73 cannot be used to justify statement (q) of the proof of the Wedgie Theorem. The reason is that Theorem 73 can only be be used after one of the statements (1) - (8) of Theorem 73 has already been proven true or false by some other means. But we know that none of those statements (1) - (8) have shown up anywhere in the earlier steps (a) - (p) of the proof of the Wedgie Theorem. Maybe we can justify statement (q) by using the Raspberry Theorem, I don't know. But we cannot use Theorem 73.

On the other hand, Theorem 73 can be used to justify statement (r) of the proof of the Wedgie Theorem. But when we do use Theorem 73 to justify statement (r), we should be very clear about what part of Theorem 73 we are using. We should not write the justification as "by Theorem 73 (2)". That would be misleading, because Theorem 73 does not tell us that (2) is true. Rather, Theorem 73 tells us only that if one of the other statements on the list is known to be true, then (2) is true as well. In our case, we are using the part of Theorem 73 that says that if (4) is known to be true, then (2) is true as well. That is, we are using Theorem 73 (4) ➔ (2).

In summary, the justifications for the steps in the fragment of the proof of the Wedgie Theorem could look like the following:

Proof of the Wedgie Theorem, with justifications

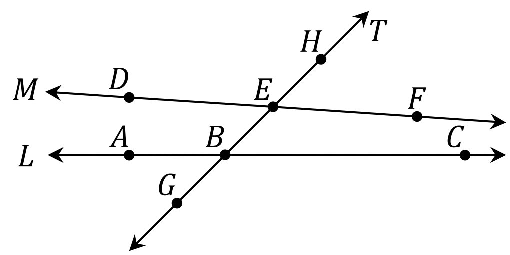
(*) some statements here

(q) $\angle ABH \cong \angle DEH$ (by Raspberry Theorem)

(r) $\angle CBE \cong \angle DEB$ (by Theorem 73 (4) ➔ (2))

(*) some more statements here

End of proof



End of Remark about "Using" an Equivalence Theorem

Since all eight of the statements above are equivalent, it makes sense to have a name for the situation in which the statements are true.

Definition 62 special angle property for two lines and a transversal

Words: Lines L and M and transversal T have the special angle property.

Meaning: The eight statements listed in Theorem 73 are true. That is, each pair of alternate interior angles is congruent. And each pair of corresponding angles is congruent. And each pair of interior angles on the same side of the transversal has measures that add up to 180.

Observe that if it is known that any one of the eight statements is true, then it is known that all eight statements are true, and so lines L and M and transversal T have the special angle property. On the other hand, if it is known that any one of the eight statements is not true, then lines L and M and transversal T do not have the special angle property. And of course in this case, it is known that all eight statements are not true.

With the notation presented above, we are prepared to understand the statement of the Alternate Interior Angle Theorem for Neutral Geometry. I have included the statement of the contrapositive because I find that for this theorem, the proof of the contrapositive is the clearest proof.

Theorem 74 The Alternate Interior Angle Theorem for Neutral Geometry

Given: Neutral Geometry, lines L and M and a transversal T

Claim: If a pair of alternate interior angles is congruent, then lines L and M are parallel.

Contrapositive: If L and M are not parallel, then a pair of alternate interior angles are not congruent.

Proof (Indirect proof by method of contraposition)

(1) Suppose that in Neutral Geometry, lines L and M and a transversal T are given, and that L and M are not parallel. **(make a drawing)** Let A be the point of intersection of lines L and M, let B be the point of intersection of lines L and T, and let C be the point of intersection of lines M and T. **(update drawing)**

(2) There exists a point D such that $A * B * D$. **(Justify.) (Make a new drawing)**
(3) Observe that $\angle CBD$ is an exterior angle for ΔABC, and $\angle BCA$ is one of its remote interior angles. **(Make a new drawing)**
(4) $m(\angle CBD) > m(\angle BCA)$. **(Justify.) (Make a new drawing)**
(5) Observe $\angle CBD$ and $\angle BCA$ are alternate interior angles and they are not congruent. That is, lines L, M, T do not have the special angle property.

End of Proof

In some geometry books, you will find three more theorems that are really corollaries of the Alternate Interior Angle Theorem. Here they are.

Three Corollaries of the Alternate Interior Angle Theorem found in some books.
Given: Neutral Geometry, lines L and M and a transversal T
Corollary (1): If a pair of corresponding angles is congruent, then lines L and M are parallel.
Corollary (2): If the sum of the measures of a pair of interior angles on the same side of the transversal is 180, then lines L and M are parallel.
Corollary (3): If L and M are both perpendicular to T, then lines L and M are parallel.
I think it is simpler to present a single corollary that uses the terminology of the special angle property.

Theorem 75 Corollary of The Alternate Interior Angle Theorem for Neutral Geometry
Given: Neutral Geometry, lines L and M and a transversal T
Claim: If any of the statements of Theorem 73 are true (that is, if lines L, M, T have the special angle property), then L and M are parallel.
Contrapositive: If L and M are not parallel, then all of the statements of Theorem 73 are false (that is, lines L, M, T do not have the special angle property).

Recall that in Section 2.1.5, we discussed the following recurring questions about parallel lines:
(1) Do parallel lines exist?
(2) Given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L?

The Alternate Interior Angle Theorem can be used to prove the following theorem that answers question (2) for Neutral Geometry.

Theorem 76 Existence of a parallel through a point P not on a line L in Neutral Geometry.
In Neutral Geometry, for any line L and any point P not on L, there exists *at least one* line M that passes through P and is parallel to L.

You will prove Theorem 76 in a homework exercise.

The Alternate Interior Angle Theorem enabled us to prove Theorem 76. That fact alone is enough to qualify the Alternate Interior Angle Theorem as a major theorem. But we will find that the theorem is used frequently throughout the rest of this book.

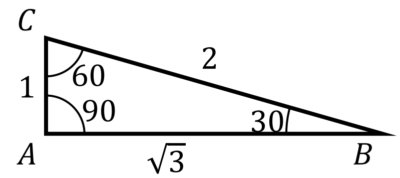
7.8. Exercises for Chapter 7

Exercises for Section 7.1 The Concept of Triangle Congruence (Section starts on page 157)

[1] Prove Theorem 51 (triangle congruence is an equivalence relation) (found on page 160)

[2] (a) What theorem or definition tells us that every triangle is congruent to itself? Explain.
(b) Given any non-collinear points A, B, C, the two symbols ΔABC and ΔACB represent the same triangle. Why?

(c) Consider the triangle shown at right. Why is the symbol $\Delta ABC \cong \Delta ACB$ not true in this case? Does that mean that you have to change your answer to (a) or (b), or is there some other explanation? Explain.



[3] What does "CPCTC" mean? Is it an axiom or a theorem, and if so, which one is it? If it is not an axiom or a theorem, then what is it? Is it even true? Explain.

[4] Here are three theorem statements:

- If the car is rusty then the apples are ripe.
- If the apples are ripe then the cow is brown.
- If the cow is brown then the car is rusty.

(A) Here are the names for the theorems, but that the names have been shuffled. Write the correct theorem statement next to each theorem name:

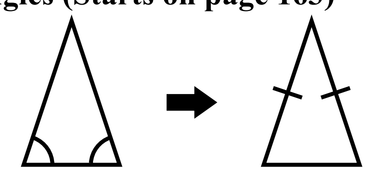
- The Brown Cow Theorem: ____
- The Rusty Car Theorem: ____
- The Ripe Apple Theorem: ____

(B) Here is statement (17) from a long and difficult proof. The earlier steps have been done correctly, but the author forgot to write the justification for statement (17). Write it in.

(17) Therefore, the car is rusty. (justification: ______________________________)

Exercises for Section 7.2 Theorems about Congruences in Triangles (Starts on page 163)

[5] Write the statement of the theorem that is illustrated by the drawing at right.



[6] Prove Theorem 53 ((Corollary) In Neutral Geometry, if a triangle is equilateral then it is equiangular.) (found on page 165)

[7] Justify the steps in the proof of Theorem 54 (the *ASA* Congruence Theorem for Neutral Geometry) (found on page 165).

[8] In the previous chapter, we discussed Theorem 40 (Every angle has a unique bisector.) (found on page 143). In order to prove it in the previous chapter, we had to first prove Theorem 39 (about points in the interior of angles) (found on page 141). Together, the proofs of those two theorems amounted to a rather difficult proof of the existence and uniqueness of angle bisectors.

I promised that there would be an easier proof that used triangle congruence. With Neutral Axiom <N10> (SAS Congruence) and Theorem 52 (CS ➔ CA) now available as tools to prove theorems, an easier proof of the existence and uniqueness of angle bisectors is possible. Justify the steps in the following proof. You may use any of the ten Neutral Axioms, and you may use any theorem up through (and including) Theorem 52 (CS ➔ CA). (Of course, you may not use Theorem 40, because that is what we are trying to prove. Re-proving earlier theorems using later theorems is actually a rather tricky business, because one must be sure to not cite any theorems whose proofs depended on the earlier theorem. But in this case, we are okay: None of the theorems between Theorem 40 and Theorem 52 depended on Theorem 40. So we can use any of them.)

Easier Proof of the Existence and Uniqueness of an Angle Bisector.

(1) Suppose that angle $\angle ABC$ is given. **(Make a drawing.)**

Part 1: Show that a bisector exists

(2) There exists a point D on ray $\overrightarrow{BC}$ such that $\overline{BD} \cong \overline{BA}$. **(Justify.) (Make a new drawing.)**
(3) $\angle BAD \cong \angle BDA$. **(Justify.) (Make a new drawing.)**
(4) There exists exactly one point M that is the midpoint of segment $\overline{AD}$. **(Justify.) (Make a new drawing.)**
(5) $\Delta BAM \cong \Delta BDM$. **(Justify.) (Make a new drawing.)**
(6) $\angle ABM \cong \angle DBM$. **(Justify.) (Make a new drawing.)**
(7) Point M is in the interior of $\angle ABD$. **(Justify.) (Make a new drawing.)**
(8) Ray $\overrightarrow{BM}$ is a bisector of angle $\angle ABD$. **(Justify.) (Make a new drawing.)**

Part 2: Show that the bisector is unique.

(9) Suppose that ray $\overrightarrow{BN}$ is a bisector of angle $\angle ABD$. **(Make a new drawing.)**
(10) $\angle ABN \cong \angle DBN$ and point N is in the interior of angle $\angle ABD$. **(Justify.) (Make a new drawing.)**
(11) Ray $\overrightarrow{BN}$ intersects side $\overline{AD}$ at a point P between A and D. **(Justify.) (Make a new drawing.)**
(12) $\Delta ABP \cong \Delta DBP$. **(Justify.) (Make a new drawing.)**
(13) $\overline{AP} \cong \overline{BP}$. **(Justify.) (Make a new drawing.)**
(14) Point P is a midpoint of segment $\overline{AD}$. **(Justify.) (Make a new drawing.)**
(15) Point P must be the same point as point M. **(Justify.) (Make a new drawing.)**
(16) Ray $\overrightarrow{BN}$ must be the same ray as ray $\overrightarrow{BM}$. **(Justify.) (Make a new drawing.)**

End of proof

[9] Prove Theorem 55 (the *CA* ➔ *CS* theorem for triangles in Neutral Geometry) (found on page 167)

Hint: Look back at the proof of Theorem 52 (the *CS* ➔ *CA* theorem for triangles (the Isosceles Triangle Theorem), found on page 164). That proof used the *SAS* axiom and a trick involving a correspondence between the vertices of a single triangle. Here is the structure of that proof:

Summary of Proof of Theorem 52

Step (1) Given information about congruent sides
Step (2) Introduce a correspondence between the vertices of ΔABC and ΔACB.

Step (3) State that the correspondence is actually a congruence (justified by the SAS Congruence Axiom <N10>)
Step (4) Conclude that the opposite angles are also congruent.
End of proof

A similar trick can be used to prove Theorem 55, the *CA* ➔ *CS* theorem for triangles. This time, the proof will use the trick involving a correspondence between the vertices of a single triangle, but the proof will cite the *ASA* Congruence Theorem, instead of the *SAS* axiom.

Structure that you should use for your proof of Theorem 55
Step (1) Given information about congruent angles
Step (2) Introduce a correspondence between the vertices of ΔABC and ΔACB.
Step (3) State that the correspondence is actually a congruence (justified by the *ASA* Congruence Theorem)
Step (4) Conclude that the opposite sides are also congruent.
End of proof

[10] Prove Theorem 56 ((Corollary) In Neutral Geometry, if a triangle is equiangular then it is equilateral.) (found on page 167)

[11] (Advanced) Justify the given steps and fill in the missing steps in the proof of Theorem 58 (the *SSS* congruence theorem for Neutral Geometry) (found on page 168)

Exercises for Section 7.3 Theorems about Bigger and Smaller Parts of Triangles (p. 170)

[12] Justify the steps in the proof of Theorem 59 (Neutral Exterior Angle Theorem) (found on page 170)

[13] Prove Theorem 60 ((Corollary) If a triangle has a right angle, then the other two angles are acute.) (found on page 172)

[14] Justify the steps in the proof of Theorem 61 (the *BS* ➔ *BA* theorem for triangles in Neutral Geometry) (found on page 172)

[15] Justify the steps in the proof of Theorem 64 (The Triangle Inequality for Neutral Geometry) (found on page 174)

Exercises for Section 7.5 More About Perpendicular Lines (Section starts on page 178)

[16] Justify the steps in the proof of Theorem 66 (existence and uniqueness of a line that is perpendicular to a given line through a given point that *does not lie on* the given line) (found on page 178)

[17] Justify the steps in the proof of Theorem 67 (The shortest segment connecting a point to a line is the perpendicular.) (found on page 180)

[18] Prove Theorem 68 (In any right triangle in Neutral Geometry, the hypotenuse is the longest side.) (found on page 181) Here are some hints:

- Use the following proof structure: Given any ΔABC with right angle at A, show that hypotenuse $\overline{BC}$ is longer than leg $\overline{AB}$ and also longer than leg $\overline{AC}$.

- Somewhere in your proof, use Theorem 60 and Theorem 62.

[19] (Advanced) Justify the steps in the proof of Theorem 69 ((Lemma) In any triangle in Neutral Geometry, the altitude to a longest side intersects the longest side at a point between the endpoints.) (found on page 181) Make drawings where indicated, and fill in the missing details where indicated.

Exercises for Section 7.6 A Final Look at Triangle Congruence in Neutral Geometry (Section starts on page 182)

[20] (Advanced) Justify the steps in the proof of Theorem 70 (the Angle-Angle-Side (*AAS*) Congruence Theorem for Neutral Geometry) (found on page 183) Make drawings where indicated, and fill in the missing details where indicated.

[21] Here is avery short proof of The Angle-Angle-Side Congruence Theorem from the internet.

Proof found on the internet for the AAS Congruence Theorem.

(1) Suppose that in triangles ΔABC and ΔDEF have $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$.

(2) We can do the following computations:

$$\begin{aligned} m(\angle C) &= 180 - \big(m(\angle A) + m(\angle B)\big) \ \ (because\ angles\ add\ up\ to\ 180) \\ &= 180 - \big(m(\angle D) + m(\angle E)\big) \ \ (because\ \angle A \cong \angle D\ and\ \angle B \cong \angle E) \\ &= m(\angle F)\ (because\ angles\ add\ up\ to\ 180) \end{aligned}$$

Therefore, we know that $\angle C \cong \angle F$.

(3) $\Delta ABC \cong \Delta DEF$. (by steps (1) and (2) and the *ASA* congruence theorem.)

End of Proof

Why can't we use this proof to prove our Theorem 70 (the Angle-Angle-Side (*AAS*) Congruence Theorem for Neutral Geometry) (found on page 183)?

[22] Justify the steps in the proof of Theorem 71 (the Hypotenuse Leg Congruence Theorem for Neutral Geometry) (found on page 183) Make drawings where indicated, and fill in the missing details where indicated.

[23] Here is a very simple proof of The Hypotenuse-Leg Congruence Theorem from the internet.

Proof found on the internet for the Hypotenuse-Leg Congruence Theorem.

(1) Suppose that right triangles ΔABC and ΔDEF have right angles at $\angle A$ and $\angle D$, and congruent hypotenuses $\overline{BC} \cong \overline{EF}$, and a congruent leg $\overline{AB} \cong \overline{DE}$,

(2) We can do the following computations:

$$\begin{aligned} (AC)^2 &= (BC)^2 - (AB)^2 \quad (Pythagorean\ Theorem) \\ &= (EF)^2 - (DE)^2 \quad (since\ \overline{BC} \cong \overline{EF}\ and\ \overline{AB} \cong \overline{DE}) \\ &= (DF)^2 \qquad\qquad\quad (Pythagorean\ Theorem) \end{aligned}$$

Therefore, we know that $\overline{AC} \cong \overline{DF}$.

(3) $\Delta ABC \cong \Delta DEF$. (by steps (1) and (2) and the *SSS* congruence theorem.)

End of Proof

Why can't we use this proof to prove our Theorem 71 (the Hypotenuse Leg Congruence Theorem for Neutral Geometry) (found on page 183)?

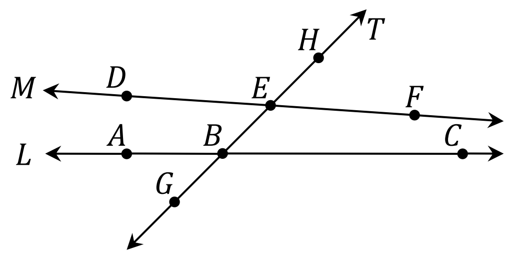
[24] (Advanced) The statement of Theorem 72, the Hinge Theorem for Neutral Geometry, is found on page 185. Find a proof of the theorem on the internet or in another book. Rewrite the proof in the style of our proofs. That is, produce a numbered list of statements, with each statement justified by a prior statement, or a prior theorem (from this book!) or a Neutral Geometry Axiom.

Exercises for Section 7.7 Parallel lines in Neutral Geometry (Section starts on page 185)

[25] Prove Theorem 73 (Equivalent statements about angles formed by two lines and a transversal in Neutral Geometry) (found on page 186). Include drawings with your proof.

[26] Consider the following fragment of the proof of the Bobcat Theorem. Its steps are lettered, rather than numbered.

Proof of the Bobcat Theorem
(*) some statements here
(x) $m(\angle ABE) + m(\angle DEB) = 180$
(y) $\angle ABG \cong \angle DEG$
(*) some more statements here
End of proof



Assume that none of the statements (1) - (8) from Theorem 73 have shown up anywhere in the earlier steps (a) - (w) of the proof of the Bobcat Theorem. Here are two questions:
(i) Why can't Theorem 73 be used to justify step (x)? Explain.
(ii) Write a justification for step (y).

[27] Justify the steps in the proof of Theorem 74 (The Alternate Interior Angle Theorem for Neutral Geometry) (found on page 188). Make drawings where indicated.

[28] Prove Theorem 76 (Existence of a parallel through a point P not on a line L in Neutral Geometry.) (found on page 189).
Hints:

- Be sure to use the correct proof structure. The given information must go into step (1).
- Show that there exists a line T that passes through P and is perpendicular to L.
- Show that there exists a line M that passes through P and is perpendicular to T.
- Show that lines L and M are parallel.

[29] In Section 2.1.5, we discussed the following recurring questions about parallel lines:

(1) Do parallel lines exist?
(2) Given a line L and a point P that does not lie on L, how many lines exist that pass through P and are parallel to L?

In Section 7.7, it was mentioned that in Neutral Geometry, Theorem 76 (Existence of a parallel through a point P not on a line L in Neutral Geometry.) (found on page 189) gives us the answer to question (2): There is at least one line. It seems that in Neutral Geometry, the answer to question (1) ought to be "yes". But can you prove that parallel lines exist in Neutral Geometry? That is, can you prove that there exist lines J and K that are parallel? Try it.

8. Neutral Geometry VI: Circles

This will be our final chapter on Neutral Geometry. In previous chapters, we discussed the behavior of lines, angles, and triangles in Neutral Geometry. Note that a line is a primitive, undefined object, but angles and triangles are defined objects. Even so, all three objects are at least mentioned in the axioms for Neutral Geometry. In this chapter, we will define circles and study their behavior. Here is the definition:

Definition 63 circle, center, radius, radial segment, interior, exterior

Symbol: $Circle(P, r)$

Spoken: the circle centered at P with radius r

Usage: P is a point and r is a positive real number.

Meaning: The following set of points: $Circle(P, r) = \{Q \text{ } such \text{ } that \text{ } PQ = r\}$

Additional Terminology:

- The point P is called the *center* of the circle.
- The number r is called the *radius* of the circle.
- The *interior* is the set $Interior\big(Circle(P, r)\big) = \{Q \text{ } such \text{ } that \text{ } PQ < r\}$.
- The *exterior* is the set $Exterior\big(Circle(P, r)\big) = \{Q \text{ } such \text{ } that \text{ } PQ > r\}$.
- Two circles are said to be *congruent* if they have the same radius.
- Two circles are said to be *concentric* if they have the same center.

Observe that the Neutral Geometry axioms do not even mention circles. So everything that we know about the behavior of circles will have to be proven in theorems.

8.1. Theorems about Lines Intersecting Circles

Our first investigations of circles will involve their interactions with objects that we already know about: lines, angles, triangles. We start with a simple but important theorem about the number of possible intersections of a line and a circle.

Theorem 77 In Neutral Geometry, the Number of Possible Intersection Points for a Line and a Circle is 0, 1, 2.

The proof is left as an exercise for readers interested in advanced topics and for graduate students.

We have special names for lines that intersect a circle at exactly one point and lines that intersect a circle at exactly two points.

Definition 64 Tangent Line and Secant Line for a Circle

A tangent line for a circle is a line that intersects the circle at exactly one point.

A secant line for a circle is a line that intersects the circle at exactly two points.

The remaining theorems in this section give us more details about the interactions of tangent lines, secant lines, and circles. The first one is an equivalence theorem. The proof structure is interesting because it uses the contrapositive. You will justify the proof in an exercise

Theorem 78 In Neutral Geometry, tangent lines are perpendicular to the radial segment.

Given: A segment $\overline{AB}$ and a line L passing through point B.

Claim: The following statements are equivalent.

(i) Line L is perpendicular to segment $\overline{AB}$.

(ii) Line L is tangent to $Circle(A, AB)$ at point B. That is, L only intersects $Circle(A, AB)$ at point B.

Proof that (i) ➔ (ii)

(1) Suppose that (i) is true. That is, suppose that line L is perpendicular to segment $\overline{AB}$. **(Make a drawing.)**

(2) Let C be any point on line L except B. **(Make a new drawing.)**

(3) Observe that points A, B, C form a triangle with right angle at B. Segment $\overline{AC}$ is the hypotenuse of this triangle. So $AC > AB$. Therefore C is in the exterior of $Circle(A, AB)$. **(Make a new drawing.)**

(4) Conclude that line L intersects $Circle(A, AB)$ at point B, but does not intersect the circle at any other point. That is, line L is tangent to $Circle(A, AB)$ at point B. So (ii) is true.

Proof that ~(i) ➔ ~(ii) (Notice that this is the contrapositive of (ii) ➔ (i).)

(5) Suppose that (i) is false. That is, suppose that line L is not perpendicular to segment $\overline{AB}$.

(6) There are two possibilities:

(a) Line L is the same line as line $\overleftrightarrow{AB}$.

(b) Line L is not the same line as line $\overleftrightarrow{AB}$.

Case (a)

(7) Suppose that line L is the same line as line $\overleftrightarrow{AB}$. **(Make a new drawing.)**

(8) There exists a point C such that $C * A * B$ and such that $\overline{AC} \cong \overline{AB}$. **(Justify) (Make a new drawing.)**

(9) Points B and C are both on $Circle(A, AB)$. **(Justify.) (Make a new Drawing.)**

(10) Conclude that line L intersects $Circle(A, AB)$ at more than one point. That is, line L is not tangent to $Circle(A, AB)$. So, Statement (ii) is false in this case.

Case (b)

(11) Suppose that line L passes through point B and is not perpendicular to line $\overleftrightarrow{AB}$ and is not the same line as line $\overleftrightarrow{AB}$. **(Make a new drawing.)**

(12) There exists a line M that passes through point A and is perpendicular to L. Let C be the point of intersection of lines L and M. **(Make a new drawing.)**

(13) There exists a point D such that $B * C * D$ and $\overline{CD} \cong \overline{CB}$. **(Make a new drawing.) (Justify.)**

(14) Observe that $\Delta ACD \cong \Delta ACB$. **(Justify.)**

(15) Therefore, $\overline{AD} \cong \overline{AB}$. **(Justify.)** So point D also lies on the circle. **(Make a new drawing.)**

(16) Conclude that L intersects the circle at more than one point. That is, line L is not tangent to the circle. So Statement (ii) is false in this case as well.

Conclusion of Cases

(17) Conclude that statement (ii) is false in either case.

End of Proof

The following corollary is not difficult to prove. You will prove it in an exercise.

Theorem 79 (Corollary of Theorem 78) For any line tangent to a circle in Neutral Geometry, all points on the line except for the point of tangency lie in the circle's exterior.

We are so used to the simple behavior of lines and circles in drawings that it is a bit of a shock (and a nuisance) to realize that we have to prove that the same simple behavior occurs in axiomatic geometry. It is even more annoying to find that the proofs are sometimes hard. The following theorem is not terribly hard to prove, but there are a lot of details. You will justify it in a homework exercise.

Theorem 80 about points on a Secant line lying in the interior or exterior in Neutral Geometry

Given: $Circle(A, r)$ and a secant line L passing through points B and C on the circle

Claim:

(i) If $B * D * C$, then D is in the interior of the circle.

(ii) If $D * B * C$ or $B * C * D$, then D is in the exterior of the circle.

Proof (for readers interested in advanced topics and for graduate students)

(1) In Neutral Geometry, suppose that $Circle(A, r)$ and a secant line L passing through points B and C on the circle are given.

(2) Observe that $\overline{AB} \cong \overline{AC}$, so ΔABC is isosceles.

(3) Therefore, $\angle ABC \cong \angle ACB$. **(Justify.)**

Part 1: prove that (i) is true.

(4) Suppose that $B * D * C$.

(5) Observe that $\angle ADB$ is an exterior angle for triangle ΔADC, and angle $\angle ACB$ is one of its remote interior angles.

(6) Therefore, $m(\angle ADB) > m(\angle ACB)$. **(Justify.)**

(7) So $m(\angle ADB) > m(\angle ABC)$. **(Justify.)**

(8) Therefore, $AB > AD$. **(Justify.)**

(9) So $AD < r$. Conclude that D is in the interior of the circle.

Part 2: prove that (ii) is true.

(10) Suppose that $D * B * C$.

(11) Observe that $\angle ABC$ is an exterior angle for triangle ΔADB, and angle $\angle ADC$ is one of its remote interior angles.

(12) Therefore, $m(\angle ABC) > m(\angle ADB)$. **(Justify.)**

(13) So $m(\angle ACB) > m(\angle ADB)$. **(Justify.)**

(14) Therefore, $AD > AC$. **(Justify.)**

(15) So $AD > r$. Conclude that D is in the exterior of the circle.

(16) – (21) Six steps analogous to steps (10) – (15) would prove that if $B * C * D$ then D is also in the exterior of the circle.

End of Proof

Now that we have determined which points on secant lines lie in the interior of a circle and which ones lie in the exterior, we can state the definition of a chord of a circle and then state an immediate corollary about the points of a chord lying in the interior of the circle.

Definition 65 chord, diameter segment, diameter, radial segment

- A *chord* of a circle is a line segment whose endpoints both lie on the circle.
- A *diameter segment* for a circle is a chord that passes through the center of the circle.

- The *diameter* of a circle is the number $d = 2r$. That is, the diameter is the number that is the length of a diameter segment.
- A *radial segment* for a circle is a segment that has one endpoint at the center of the circle and the other endpoint on the circle. (So that the radius is the number that is the length of a radial segment.)

Theorem 81 (Corollary of Theorem 80) about points on a chord or radial segment that lie in the interior in Neutral Geometry

In Neutral Geometry, all points of a segment except the endpoints lie in the interior of the circle. Furthermore, one endpoint of a radial segment lies on the circle; all the other points of a radial segment lie in the interior of the circle.

As mentioned above, it is a little frustrating to have to prove that abstract circles have behavior that seems so simple and obvious in drawn circles. The next two theorems have statements that are obvious. You will justify the steps in the proof of the first theorem in a homework exercise.

Theorem 82 In Neutral Geometry, if a line passes through a point in the interior of a circle, then it also passes through a point in the exterior.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that a line L passes through a point B in the interior of $Circle(A,r)$.
(2) There exists a point C on line L such that $BC = 3r$.
(3) $BC \leq BA + AC$ **(Justify.)**
(4) $BC - BA \leq AC$ **(arithmetic)**
(5) $r > BA$ **(Justify.)**
(6) $-r < -BA$ **(arithmetic)**
(7) $2r = 3r - r < 3r - BA = BC - BA$ **(arithmetic)**
(8) $2r \leq AC$ **(Justify.)**
(9) Therefore, point C is in the exterior.

End of Proof

Theorem 83 In Neutral Geometry, if a line passes through a point in the interior of a circle and also through a point in the exterior, then it intersects the circle at a point between those two points.

As simple Theorem 83 sounds, it is difficult to prove. One proof involves defining a new function involving a coordinate function f for the line and also the distance function d. We will accept the theorem without proof in this book. The following corollary is an example of how the theorem is used.

Theorem 84 (Corollary) In Neutral Geometry, if a line passes through a point in the interior of a circle, then the line must be a secant line. That is, the line must intersect the circle exactly twice.

Proof

Suppose a line passes through a point in the interior of a circle. By Theorem 82, we know that the line must also pass through a point in the exterior. By Theorem 83, we know that the line must intersect the circle. By Theorem 79, we know that the line cannot be a tangent line because it passes through an interior point. Therefore, the line must be a secant line, and intersect the circle exactly twice.

End of Proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 8.6 on page 205.

8.2. A Digression: Two Theorems About Triangles

Now that we have defined chords, our goal is to prove some basic theorems about their behavior. But before doing that, we should first study two theorems about triangles. The reason is that we will find that the techniques used in the proofs of the theorems about chords will be the same techniques that are used in the proofs of these theorems about triangles. These two theorems about triangles could have been stated earlier in the book. I postponed introducing them until now because I did not want the earlier chapters to be overfull and also because I thought it would be nice to present the theorems here, where we can see them put to immediate use.

The first theorem has to do with angle bisectors, altitudes, and medians in isosceles triangles in Neutral Geometry. Altitudes were introduced in Definition 59. We need a definition of medians.

Definition 66 median segment and median line of a triangle

A *median line* for a triangle is a line that passes through a vertex and the midpoint of the opposite side. A *median segment* for triangle is a segment that has its endpoints at those points.

In general, for a given vertex of a scalene triangle in Neutral Geometry, the angle bisector, altitude, and median drawn from that vertex will be three different rays. The following theorem tells us that in the special case of the top vertex of an isosceles triangle, those three special rays are in fact the same ray. You will justify the steps and supply some missing steps in the proof in a homework exercise.

Theorem 85 about special rays in isosceles triangles in Neutral Geometry

Given: Neutral Geometry, triangle ΔABC with $\overline{AB} \cong \overline{AC}$ and ray $\overrightarrow{AD}$ such that $B * D * C$.

Claim: The following three statements are equivalent.

(i) Ray $\overrightarrow{AD}$ is the bisector of angle $\angle BAC$.

(ii) Ray $\overrightarrow{AD}$ is perpendicular to side $\overline{BC}$.

(iii) Ray $\overrightarrow{AD}$ bisects side $\overline{BC}$. That is, point D is the midpoint of side $\overline{BC}$.

Proof that (i) → (ii)

(1) Suppose that (i) is true. That is, suppose that ray $\overrightarrow{AD}$ is the bisector of angle $\angle BAC$. **(Make a drawing.)**

(2) $\Delta ADB \cong \Delta ADC$. **(Justify.)**

(3) Therefore, $\angle ADB \cong \angle ADC$. **(Justify.)**

(4) So $m(\angle ADB) = 90$. **(Justify.)**

(5) Therefore, ray $\overrightarrow{AD}$ is perpendicular to side $\overline{BC}$. That is, statement (ii) is true.

Proof that (ii) → (iii)

(6) Suppose that (ii) is true. That is, suppose that ray $\overrightarrow{AD}$ is perpendicular to side $\overline{BC}$. **(Make a drawing.)**

(7) $\Delta ADB \cong \Delta ADC$. **(Justify.)**

(8) Therefore, $\overline{DB} \cong \overline{DC}$. **(Justify.)**
(9) So point D is the midpoint of side $\overline{BC}$. That is, statement (iii) is true.

Proof that (iii) ➔ (i)
(*) You supply the missing steps.

End of Proof

The next theorem has a rather quirky-sounding statement. It turns out to be an exteremely useful theorem. The proof, which is left as an exercise, will involve techniques similar to the techniques used in the proof of Theorem 85.

Theorem 86 about points equidistant from the endpoints of a line segment in Neutral Geometry
In Neutral Geometry, the following two statements are equivalent
(i) A point is equidistant from the endpoints of a line segment.
(ii) The point lies on the perpendicular bisector of the segment.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 8.6 on page 205.

8.3. Theorems About Chords

With Theorem 85 at our disposal, we now go back to our job of proving theorems about chords. Our first two theorems have simple proofs that pretty much use only Theorem 85.

Theorem 87 In Neutral Geometry, any perpendicular from the center of a circle to a chord bisects the chord

Proof
(1) In Neutral Geometry, in $Circle(A, r)$, suppose that line L passes through the center and is perpendicular to chord $\overline{BC}$. Let D be the point of intersection. **(Make a drawing.)**
(2) Observe observe that ΔABC has $\overline{AB} \cong \overline{AC}$. **(Make a new drawing.)**
(3) By Theorem 85, we know that point D is the midpoint of side $\overline{BC}$. **(Make a new drawing.)**

End of Proof

Theorem 88 In Neutral Geometry, the segment joining the center to the midpoint of a chord is perpendicular to the chord.

The proof of Theorem 88 is left to the reader. (The proof is very much like the proof of the previous theorem.)

Our third theorem about chords is really just a corollary of Theorem 86.

Theorem 89 (Corollary of Theorem 86) In Neutral Geometry, the perpendicular bisector of a chord passes through the center of the circle.

The above three theorems about chords had short proofs involving Theorem 85 or Theorem 86. Our fourth theorem about chords has a longer proof. You will justify some of the steps and make drawings for it in a homework exercise.

Theorem 90 about chords equidistant from the centers of circles in Neutral Geometry

Given: Neutral Geometry, chord $\overline{BC}$ in $Circle(A,r)$ and chord $\overline{QR}$ in $Circle(P,r)$ with the same radius r.

Claim: The following two statements are equivalent:

(i) The distance from chord $\overline{BC}$ to center A is the same as the distance from chord $\overline{QR}$ to center P.

(ii) The chords have the same length. That is, $\overline{BC} \cong \overline{QR}$.

Proof that (i) ➔ (ii)

(1) Suppose that (i) is true. That is, the distance from chord $\overline{BC}$ to center A is the same as the distance from chord $\overline{QR}$ to center P. **(Make a drawing.)**

(2) Let line L be the line that passes through point A and is perpendicular to chord $\overline{BC}$ in $Circle(A,r)$. Let D be the point of intersection. And let line M be the line that passes through point P and is perpendicular to chord $\overline{QR}$ in $Circle(P,r)$. Let S be the point of intersection. **(Make a new drawing.)**

(3) $\overline{AD} \cong \overline{PS}$. (by statements (1), (2)) **(Make a new drawing.)**

(4) Observe that triangles ΔABD and ΔPQS have right angles at D and S, and congruent hypotenuses $\overline{AB} \cong \overline{PQ}$ (because the circles have the same radius r) and congruent legs $\overline{AD} \cong \overline{PS}$. (by (3))

(5) Therefore, $\Delta ABD \cong \Delta PQS$. **(Justify.)**

(6) So $\overline{BD} \cong \overline{QS}$. (by (5) and the definition of triangle congruence) **(Make a new drawing.)**

(7) We know that point D is the midpoint of chord $\overline{BC}$ and that point S is the midpoint of chord $\overline{QR}$. **(Justify.)**

(8) So $\overline{BC} \cong \overline{QR}$. (by (6), (7) and arithmetic) That is, statement (ii) is true. **(New drawing.)**

Proof that (ii) ➔ (i)

(9) Suppose that (ii) is true. That is, suppose that $\overline{BC} \cong \overline{QR}$. **(Make a new drawing.)**

(10) Let D be the midpoint of side $\overline{BC}$, and let S be the midpoint of side $\overline{QR}$. **(New drawing.)**

(11) Then $\overline{BD} \cong \overline{QS}$. (by (9), (10))

(12) Observe that triangle ΔABC has $\overline{AB} \cong \overline{AC}$. **(Make a new drawing.)**

(13) Therefore, segment $\overline{AD}$ is perpendicular to side $\overline{BC}$. **(Justify.) (Make a new drawing.)**

(14) For the same reason, we know that segment $\overline{PS}$ is perpendicular to side $\overline{QR}$.

(15) Observe that triangles ΔABD and ΔPQS have right angles at D and S, and congruent hypotenuses $\overline{AB} \cong \overline{PQ}$ (because the circles have the same radius r) and congruent legs $\overline{BD} \cong \overline{QS}$. (by (11)) **(Make a new drawing.)**

(16) Therefore, $\Delta ABD \cong \Delta PQS$. **(Justify.)**

(17) So $\overline{AD} \cong \overline{PS}$. (by (16) and the definition of triangle congruence) That is, statement (i) is true. **(Make a new drawing.)**

End of proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 8.6 on page 205.

8.4. A Digression: Two Theorems About Angle Bisectors

We will wrap up our study of circles in Neutral Geometry by looking more at lines tangent to circles and also looking at what are called tangent circles. Before doing that, it is useful to again digress to present theorems about angles and triangles, theorems that could have been presented earlier in the book.

The first theorem we will discuss refers to the distance from a point to a line. Recall that we proved in Theorem 66 that for any line, and any point not on the line, there is exactly one line that passes through the point and is perpendicular to the original line. And we proved in Theorem 67 that the shortest segment connecting a point to a line is the perpendicular. So when one speakes of the distance from a point to a line, one is referring to the length of the segment that has one endpoint on the given point and the other endpoint on the line and that is perpendicular to the line.

Theorem 91 about points on the bisector of an angle in Neutral Geometry
Given: Neutral Geometry, angle $\angle BAC$, and point D in the interior of the angle
Claim: The following statements are equivalent
(i) D lies on the bisector of angle $\angle BAC$.
(ii) D is equidistant from the sides of angle $\angle BAC$.

Proof

(1) In Neutral Geometry, suppose that point D lies in the interior of angle $\angle BAC$. **(Make a drawing.)**

Proof that (i) ➔ (ii)

(2) Suppose that (i) is true. That is, suppose that D lies on the bisector of angle $\angle BAC$. **(Make a new drawing.)**
(3) Let point E be the foot of the perpendicular from D to line $\overleftrightarrow{AB}$, and let point F be the foot of the perpendicular from D to line $\overleftrightarrow{AC}$. **(Make a new drawing.)**
(4) Then $\Delta DAE \cong \Delta DAF$. **(Justify.) (Make a new drawing.)**
(5) So $\overline{DE} \cong \overline{DF}$. **(Justify.) (Make a new drawing.)**
(6) Conclude that D is equidistant from the sides of angle $\angle BAC$. That is, (ii) is true.

Proof that (ii) ➔ (i)

(7) Suppose that (ii) is true. That is, suppose that D is equidistant from the sides of angle $\angle BAC$. **(Make a new drawing.)**
(8) Let point E be the foot of the perpendicular from D to line $\overleftrightarrow{AB}$, and let point F be the foot of the perpendicular from D to line $\overleftrightarrow{AC}$. **(Make a new drawing.)**
(9) Then $\overline{DE} \cong \overline{DF}$. **(Justify.) (Make a new drawing.)**
(10) Then $\Delta DAE \cong \Delta DAF$. **(Justify.) (Make a new drawing.)**
(11) So $\angle DAE \cong \angle DAF$. **(Justify.) (Make a new drawing.)**
(12) Conclude that D lies on the bisector of angle $\angle BAC$. That is, (i) is true.

End of Proof

Our next theorem about angle bisectors is of a type called a concurrence theorem. Remember that a collection of lines is called concurrent if there exists a point that all the lines pass through. In this book, we will discuss three concurrence theorems about special lines related to triangles. This first concurrence theorem is about angle bisectors. You will justify the steps in the proof in an exercise.

Theorem 92 in Neutral Geometry, the three angle bisectors of any triangle are concurrent at a point that is equidistant from the three sides of the triangle.

Proof

(1) In Neutral Geometry, suppose that triangle ΔABC is given. **(Make a drawing.)**

Show that the bisectors of $\angle A$ and $\angle B$ intersect.

(2) There exists a ray $\overrightarrow{AD}$ that bisects $\angle CAB$. **(Justify.) (Make a new drawing.)**
(3) Point D lies in the interior of angle $\angle CAB$. **(Justify.) (Make a new drawing.)**
(4) Ray $\overrightarrow{AD}$ intersects side $\overline{BC}$ at a point that we can call E. **(Justify.) (Make a new drawing.)**
(5) There exists a ray $\overrightarrow{BF}$ that bisects $\angle ABE$. **(Justify.) (Make a new drawing.)**
(6) Point F lies in the interior of angle $\angle ABE$. **(Justify.) (Make a new drawing.)**
(7) Ray $\overrightarrow{BF}$ intersects segment $\overline{AE}$ at a point that we can call G. **(Justify.) (Make a new drawing.)** We have shown that the bisectors of $\angle A$ and $\angle B$ intersect at G.

Consider distances from the point of intersection to the sides of the triangle

(8) The distance from G to line $\overleftrightarrow{AC}$ equals the distance from G to line $\overleftrightarrow{AB}$. **(Justify.) (Make a new drawing.)**
(9) The distance from G to line $\overleftrightarrow{BA}$ equals the distance from G to line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**
(10) So the distance from G to line $\overleftrightarrow{CA}$ equals the distance from G to line $\overleftrightarrow{CB}$. **(Justify.) (Make a new drawing.)**
(11) Therefore, point G lies on the bisector of $\angle BCA$. **(Justify.) (Make a new drawing.)**
(12) We have shown that the bisectors of all three angles $\angle A$ and $\angle B$ and $\angle C$ intersect at G and that G is equidistant from the three sides of the triangle.

End of Proof

The point where the three angle bisectors meet is called the incenter of the triangle. This name might sound strange, but the reason for it will become clearer in the next section.

Definition 67 incenter of a triangle
The incenter of a triangle in Neutral Geometry is defined to be the point where the three angle bisectors meet. (Such a point is guaranteed to exist by Theorem 92.)

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 8.6 on page 205.

8.5. Theorems About Tangent Lines and Inscribed Circles

Our first theorem about tangent lines drawn to circles has a very short proof. You will be asked to prove the theorem in an exercise.

Theorem 93 about tangent lines drawn from an exterior point to a circle in Neutral Geometry
Given: Neutral Geometry, $Circle(P,r)$, point A in the exterior of the circle, and lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ tangent to the circle at points B and C.
Claim: $\overline{AB} \cong \overline{AC}$ and $\angle PAB \cong \angle PAC$.

The next theorem proves that if three points lie on a circle, then they do not lie on any other circle. The proof mainly uses one idea, a theorem from earlier in the chapter. In a homework exercise, you will be asked to identify that theorem and to make drawings.

Theorem 94 In Neutral Geometry, if three points lie on a circle, then they do not lie on any other circle.

Proof

(1) In Neutral Geometry, suppose that three distinct points A, B, C lie on $Circle(P, r_1)$ and on $Circle(Q, r_2)$. (We will show that $P = Q$ and $r_1 = r_2$.)

Show that P lies at the intersection of two lines L and M.

(2) Observe that point P is equidistant from point A and B. **(Make a drawing.)**

(3) Therefore, point P lies on the line that is the perpendicular bisector of segment $\overline{AB}$. **(Which theorem justifies that statement?)** Call this line L. **(Make a new drawing.)**

(4) Observe that point P is equidistant from point B and C. **(Make a new drawing.)**

(5) Therefore, point P lies on the line that is the perpendicular bisector of segment $\overline{BC}$. **(Which theorem justifies that statement?)** Call this line M. **(Make a new drawing.)**

(6) So point P lies on both lines L and M. **(Make a new drawing.)**

Show that Q also lies at the intersection of L and M.

(7) Observe that point Q is equidistant from point A and B and that Q is also equidistant from point B and C. Therefore, steps identical to steps (2) – (5) would show that point Q lies on both lines L and M.

Conclusion

(8) We know that lines L and M are not the same line. (because that would require that points A and C be the same point) So the lines can only intersect in one point. Therefore, P and Q must be the same point. That is, $P = Q$.

(9) That means that $r_2 = QA = PA = r_1$.

(10) Conclude that $Circle(P, r_1)$ and $Circle(Q, r_2)$ are in fact the same circle.

End of proof

So far, we have studied situations involving one or two tangent lines on a given circle. We now turn our attention to circles inscribed in polygon. Here, there are many tangent lines on a given circle.

Definition 68 inscribed circle

An *inscribed circle* for a polygon is a circle that has the property that each of the sides of the polygon is tangent to the circle.

For a given polygon, it is not always possible to inscribe a circle in the polygon. But it turns out that it is always possible to inscribe a circle in a triangle. Here's the theorem. You will justify it in the exercises.

Theorem 95 in Neutral Geometry, every triangle has exactly one inscribed circle.

Proof

(1) In Neutral Geometry, suppose that triangle ΔABC is given. **(Make a drawing.)**

Show that an inscribed circle exists.

(2) The three angle bisectors are concurrent at a point P that is equidistant from the three sides of the triangle. **(Justify.) (Make a new drawing.)**

(3) Let D be the point on side $\overline{AB}$ such that $\overline{PD}$ is perpendicular to side $\overline{AB}$, Let E be the point on side $\overline{BC}$ such that $\overline{PE}$ is perpendicular to side $\overline{BC}$, and let F be the point on side $\overline{CA}$ such that $\overline{PF}$ is perpendicular to side $\overline{CA}$. **(Make a new drawing.)**

(4) Then $\overline{PD} \cong \overline{PE} \cong \overline{PF}$. (by (2) and (3))

(5) Let $r = PD = PE = PF$.
(6) Observe that all three points D, E, F lie on $Circle(P, r)$. (by (5)) **(Make a new drawing.)**
(7) Note that side $\overline{AB}$ is tangent to $Circle(P, r)$ at D. **(Justify.)** Similarly, side $\overline{BC}$ is tangent to $Circle(P, r)$ at E, and side $\overline{CA}$ is tangent to $Circle(P, r)$ at F.
(8) Conclude that $Circle(P, r)$ is inscribed in ΔABC.

Show that there cannot be any other inscribed circles.

(9) Suppose that $Circle(Q, r_2)$ is inscribed in ΔABC. **(Make a new drawing.)** (We will show that $Q = P$ and $r_2 = r$.) Let G, H, J be the points where $Circle(Q, r_2)$ intersects sides $\overline{AB}, \overline{BC}, \overline{CA}$, respectively.
(10) Radial segments $\overline{QG}, \overline{QH}, \overline{QJ}$ must be perpendicular to sides $\overline{AB}, \overline{BC}, \overline{CA}$, respectively. **(Justify.) (Make a new drawing.)**
(11) Conclude that point Q is equidistant from the three sides of ΔABC. **(Make a new drawing.)**
(12) Therefore, point Q must lie on all three of the angle bisectors of ΔABC. **(Justify.)**
(13) Conclude that point Q must be point P.
(14) Conclude that points G, H, J must be the same as points D, E, F, respectively. **(Justify.)**
(15) Therefore, $r_2 = QG = PD = r$.
(16) We have shown that $Q = P$ and $r_2 = r$. In other words, $Circle(Q, r_2)$ is just $Circle(P, r)$.

End of Proof

We see that the inscribed circle has its center at the point where the three angle bisectors meet. This point was given the name *incenter* back in Definition 67. Now we see why the name incenter is used: the incenter is the center of the inscribed circle.

In the exercises, you will explore tangent circles. Here is the definition.

Definition 69 tangent circles
Two circles are said to be tangent to each other if they intersect in exactly one point.

The facts that you learn about tangent circles mostly follow from things that we have already learned about lines tangent to circles. But there is one fact about circles that we have not discussed and that you will need to use. Here it is:

Theorem 96 In Neutral Geometry, if one circle passes through a point that is in the interior of another circle and also passes through a point that is in the exterior of the other circle, then the two circles intersect at exactly two points.

As simple as it sounds, this theorem is difficult to prove. We will accept the theorem without proof in this book. (Notice that this happened before, with the similar Theorem 83 about a line passing through a point in the interior of a circle and also passing through a point in the exterior of the circle.) (In some books, the statement of Theorem 96 is actually an axiom.)

8.6. Exercises for Chapter 8

Exercises for Section 8.1 Theorems about Lines Intersecting Circles (Starts on page 195)

[1] (Advanced) Prove Theorem 77 (In Neutral Geometry, the Number of Possible Intersection Points for a Line and a Circle is 0, 1, 2.) (found on page 195).

[2] Justify the steps in the proof of Theorem 78 (In Neutral Geometry, tangent lines are perpendicular to the radial segment.) (found on page 196). Make drawings where indicated.

[3] Prove Theorem 79: ((Corollary of Theorem 78) For any line tangent to a circle in Neutral Geometry, all points on the line except for the point of tangency lie in the circle's exterior.) (found on page 197).

[4] (Advanced) Justify the steps in the proof of Theorem 80 (about points on a Secant line lying in the interior or exterior in Neutral Geometry) (found on page 197). Make drawings to illustrate the proof, and fill in the missing steps (16) – (21).

[5] (Advanced) Justify the steps in the proof of Theorem 82 (In Neutral Geometry, if a line passes through a point in the interior of a circle, then it also passes through a point in the exterior.) (found on page 198) Make a drawing to illustrate the proof.

Exercises for Section 8.2 A Digression: Two Theorems About Triangles (Starts on p. 199)

[6] Justify the steps and supply the missing steps in the proof of Theorem 85 (about special rays in isosceles triangles in Neutral Geometry) (found on page 199). Make a drawing for each step of the proof.

[7] Prove Theorem 86 (about points equidistant from the endpoints of a line segment in Neutral Geometry) (found on page 200). Make drawings to illustrate. your proof.

Exercises for Section 8.3 Theorems About Chords (Section starts on page 200)

[8] Make drawings to illustrate the proof of Theorem 87 (In Neutral Geometry, any perpendicular from the center of a circle to a chord bisects the chord) (found on page 200).

[9] Prove Theorem 88 (In Neutral Geometry, the segment joining the center to the midpoint of a chord is perpendicular to the chord.) (found on page 200). Make drawings to illustrate your proof.

[10] Make drawings to illustrate the proof of Theorem 89 ((Corollary of Theorem 86) In Neutral Geometry, the perpendicular bisector of a chord passes through the center of the circle.) (found on page 200). Find the theorem needed to justify step (5) in the proof.

[11] Justify the steps in the proof of Theorem 90 (about chords equidistant from the centers of circles in Neutral Geometry) (found on page 201). Make drawings where illustrated.

Exercises for Section 8.4 A Digression: Two Theorems About Angle Bisectors (p. 201)

[12] Justify the steps in the proof of Theorem 91 (about points on the bisector of an angle in Neutral Geometry) (found on page 202). Make drawings to illustrate the two sections of the proof.

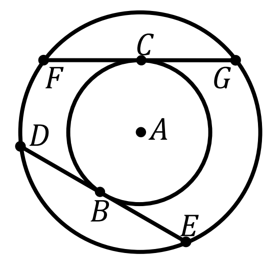
[13] Justify the steps in the proof of Theorem 92 (in Neutral Geometry, the three angle bisectors of any triangle are concurrent at a point that is equidistant from the three sides of the triangle.) (found on page 202). Make drawings to illustrate the proof.

Exercises for Section 8.5 Theorems About Tangent Lines and Inscribed Circles (p. 203)

[14] Prove Theorem 93 (about tangent lines drawn from an exterior point to a circle in Neutral Geometry) (found on page 203). Make drawings to illustrate your proof. Hint: Show first that $\Delta APB \cong \Delta APC$.

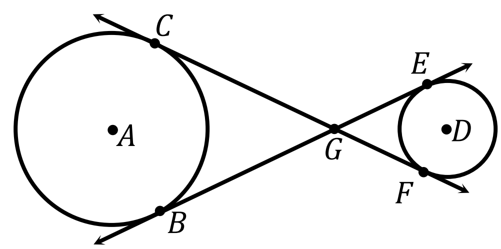
[15] Make drawings to illustrate the proof of Theorem 94 (In Neutral Geometry, if three points lie on a circle, then they do not lie on any other circle.) (found on page 204). One theorem can be used to justify statements (3), (5), (9), and (11) in the proof. What is that theorem?

[16] Justify the steps in the proof of Theorem 95 (in Neutral Geometry, every triangle has exactly one inscribed circle.) (found on page 204).Make drawings to illustrate the proof.



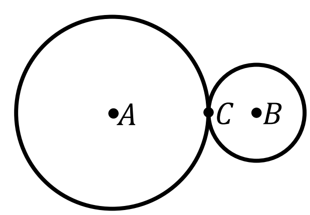
[17] Two concentric circles are centered at point A. Outer circle chords $\overline{DE}$ and $\overline{FG}$ are tangent to the inner circle at points B and C. Prove $\overline{DE} \cong \overline{FG}$.

[18] Two circles are centered at points A and D. The left circle has radius $r_1 = AB$. The right circle has radius $r_2 = DE$. Lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are tangent to the circles at points B, C, E, F, and the lines intersect at G. Prove that points A, G, D are collinear.



[19] $Circle(A, r_1)$ and $Circle(B, r_2)$ are tangent at C. Center B is in the exterior of $Circle(A, r_1)$.

(a) Prove that points A, B, C are collinear. Hint: Do an indirect proof, using the method of contradiction. Assume that C is not on $\overleftrightarrow{AB}$. Consider triangle ΔABC. It has sides of length $CA = r_1$ and $CB = r_2$. Show that there exists a point D on the other side of line $\overleftrightarrow{AB}$ such that $\Delta ABC \cong \Delta ABD$. (There will be a few steps involved.) Show that point D lies on both circles. Explain the contradiction.

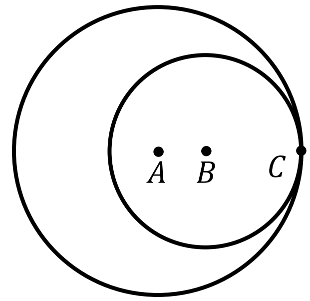


(b) Prove that the two circles have a common tangent line at point C.

[20] $Circle(A, r_1)$ and $Circle(B, r_2)$ are tangent at C. Center B is in the interior of $Circle(A, r_1)$, and radius $r_1 > r_2$.

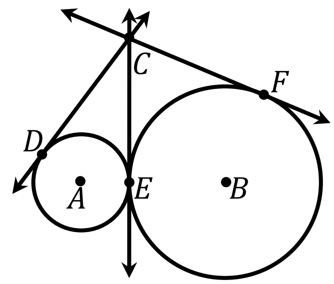
(a) Prove that the two circles have a common tangent line at point C. Hint: See the hint for [19](a). It will work here, too.

(b) Prove that the two circles have a common tangent line at point C.



[21] $Circle(A, r_1)$ and $Circle(B, r_2)$ are tangent at point E. Lines $\overleftrightarrow{CD}, \overleftrightarrow{CE}, \overleftrightarrow{CF}$ are tangent to the circles at points D, E, F.

Prove that $CD = CE = CF$.



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9.Euclidean Geometry I: Triangles

9.1. Introduction

In this section, we add an eleventh axiom to the axiom system for Neutral Geometry. The resulting longer axiom list is called the Axiom System for Euclidean Geometry.

Definition 70 The Axiom System for Euclidean Geometry

Primitive Objects: *point*, *line*

Primitive Relation: *the point lies on the line*

Axioms of Incidence and Distance

<N1> There exist two distinct points. (at least two)

<N2> For every pair of distinct points, there exists exactly one line that both points lie on.

<N3> For every line, there exists a point that does not lie on the line. (at least one)

<N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Axiom of Separation

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following three properties:

(i) The three sets L, H_1, H_2 form a partition of the set of all points.

(ii) Each of the half-planes is convex.

(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

Axioms of Angle measurement

<N7> (Angle Measurement Axiom) There exists a function $m: \mathcal{A} \to (0,180)$, called the Angle Measurement Function.

<N8> (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $m(\angle PAB) = r$.

<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Axiom of Triangle Congruence

<N10> (*SAS* Axiom) If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Euclidean Parallel Axiom

<EPA> (*EPA* Axiom) For any line L and any point P not on L, there is *not more than one* line M that passes through P and is parallel to L.

We now officially begin our study of Euclidean Geometry. However, it is important to note that because the ten Neutral Geometry Axioms are included on the list of axioms for Euclidean Geometry, every theorem of Neutral Geometry will also be a theorem of Euclidean Geometry. In this book that means that the statements of Theorems 1 through Theorem 96 are all true in Euclidean Geometry.

The wording of the eleventh postulate, the so-called *Euclidean Parallel Postulate*, is rather peculiar, and almost sounds like a mistake. But the wording is very carefully chosen to remind us of the "recurring question" in Geometry, first introduced in Section 2.1.5:

> **The recurring question in Geometry:** For any line L and any point P not on L, how many lines exist that pass through P and are parallel to L?

The Euclidean Parallel Axiom is worded the way it is to make it clear that the axiom does not guarantee us *any* lines that pass through P and are parallel to L. All the axiom says is that there cannot be more than one such line. But remember that in Neutral Geometry, we proved a theorem (Theorem 76) that says there is at least one such line. That theorem and the new Euclidean Parallel Axiom together give us a definitive answer to the recurring question in the case of Euclidean Geometry. Here is the answer, stated as a corollary. You will justify it in a homework exercise.

Theorem 97 (Corollary) In Euclidean Geometry, the answer to the recurring question is *exactly one line*.

In Euclidean Geometry, for any line L and any point P not on L, there exists *exactly one* line M that passes through P and is parallel to L.

Proof

(1) In Euclidean Geometry, suppose that line L and a point P not on L are given.
(2) There exists at least one line that passes through P and is parallel to L. **(Justify.)**
(3) There is not more than one line. **(Justify.)**
(4) Therefore, there is exactly one line.

End of Proof.

There are some other immediate corollaries of the Euclidean Parallel Axiom. Here are two. You will justify them in homework exercises.

Theorem 98 (corollary) In Euclidean Geometry, if a line intersects one of two parallel lines, then it also intersects the other.

In Euclidean Geometry, if L and M are parallel lines, and line T intersects M, then T also intersects L.

Proof

(1) In Euclidean Geometry, suppose that line L and M are parallel lines, and line T intersects M. **(Make a drawing.)**
(2) Let P be the point of intersection of line T and M.
(3) We see that line M passes through P and is parallel to line L.
(4) There cannot be a second line that passes through P and is parallel to line L. **(Justify.)**
(5) Therefore, line T cannot be parallel to line L. That is, line T must intersect line L. **(Make a new drawing.)**

End of Proof.

Theorem 99 (corollary) In Euclidean Geometry, if two distinct lines are both parallel to a third line, then the two lines are parallel to each other. That is, if distinct lines M and N are both parallel to line L, then M and N are parallel to each other.

Proof (indirect proof by contradiction)

(1) In Euclidean Geometry, suppose distinct lines M and N are both parallel to line L.
(2) Assume that M and N are not parallel to each other. **(Justify.)**
(3) Then M and N intersect at exactly one point. **(Justify.)** Let P be their point of intersection. **(Make a drawing.)**
(4) We see that lines M and N both pass through P and are parallel to line L.
(5) Statement (4) contradicts something. **(What does it contradict? There is more than one possible answer.)** So our assumption in step (2) was wrong. lines M and N must be parallel to each other.

End of Proof.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.2. Parallel Lines and Alternate Interior Angles in Euclidean Geometry

Recall that in Section 7.7, we discussed parallel lines in Neutral Geometry. We considered the situation of two lines L and M and a transversal T. Theorem 74 stated that in Neutral Geometry, if a pair of alternate interior angles is congruent, then L and M are parallel. The converse of the statement of that theorem is *not* a theorem of Neutral Geometry. But it is a theorem of Euclidean Geometry. Here is the theorem and its proof.

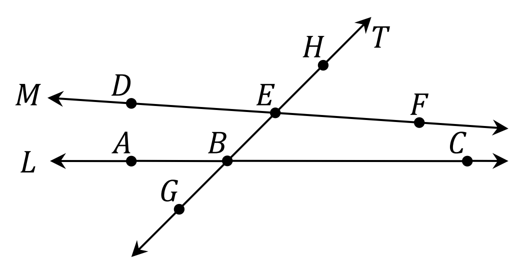
Theorem 100 Converse of the Alternate Interior Angle Theorem for Euclidean Geometry

Given: Euclidean Geometry, lines L and M and a transversal T

Claim: If L and M are parallel, then a pair of alternate interior angles is congruent.

Proof: (Method: Prove the contrapositive statement: If a pair of alternate interior angles is not congruent, then L and M are not parallel.)

(1) In Euclidean Geometry, suppose that lines L and M and a transversal T are given. Label points as in Definition 61. That is, let B be the intersection of lines T and L, and let E be the intersection of lines T and M. (By definition of transversal, B and E are not the same point.) By Theorem 15, there exist points A and C on line L such that $A * B * C$, points D and F on line M such that $D * E * F$, and points G and H on line T such that $G * B * E$ and $B * E * H$. Without loss of generality, we may assume that points D and F are labeled such that it is point D that is on the same side of line T as point A. (See the figure at right above.) And suppose that a pair of alternate interior angles is not congruent. In particular, assume that $\angle FEB \ncong \angle ABE$.



Define a new line for which alternate interior angles *are* congruent.

(2) Transversal T creates two half-planes. Let H_F be the half-plane containing F.

(3) By the Congruent Angle Construction Theorem (Theorem 48), we know that there is exactly one ray $\overrightarrow{EJ}$ with point J in H_F such that $\angle JEB \cong \angle ABE$.
(4) Consider line L and line $\overleftrightarrow{EJ}$ and transversal T. Observe that because of the way that line $\overleftrightarrow{EJ}$, was defined, this collection of lines has congruent alternate interior angles.
(5) So line $\overleftrightarrow{EJ}$ is parallel to line L. (by Theorem 74, the Alternate Interior Angle Theorem)
(6) Observe that line $\overleftrightarrow{EJ}$ passes through point E. That is, both line M and line $\overleftrightarrow{EJ}$ pass through point E.

Conclude

(7) By <EPA>, the Euclidean Parallel Axiom, there can be at most one line that passes through point E and is parallel to line L. Therefore, line M cannot be parallel to L.

End of Proof

Here are two immediate corollaries. You will prove the second one in a homework exercise.

Theorem 101 (corollary) Converse of Theorem 75.
Given: Euclidean Geometry, lines L and M and a transversal T
Claim: If L and M are parallel, then all of the statements of Theorem 73 are true (that is, lines L, M, T have the special angle property).

Theorem 102 (corollary) In Euclidean Geometry, if a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other. That is, if lines L and M are parallel, and line T is perpendicular to M, then T is also perpendicular to L.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.3. Angles of Triangles in Euclidean Geometry

You probably came into this course with rusty geometry skills. But one of the things that you probably did remember was that the angle sum for any triangle is 180. So it may have been puzzling for you that we never got to use that fact in the first eight chapters of this book. The reason that you never got to use the fact is that in Neutral Geometry, it is simply not true that the angle sum for any triangle is 180. Now that we are at last studying Euclidean Geometry, we can finally prove the fact about angle sums and start using it. Here is a proof. You will justify it in a class drill.

Theorem 103 In Euclidean Geometry, the angle sum for any triangle is 180.

Proof:

(1) In Euclidean Geometry, suppose that triangle ΔABC is given.
(2) Let L be line $\overleftrightarrow{AB}$.
(3) There exists a line M that passes through C and is parallel to L. **(Justify.) (make a drawing)**
(4) There exist points D, E on line M such that $D * C * E$. Points D and E can be labeled so that it is point D that is on the same side of line $\overleftrightarrow{BC}$ as point A.
(5) $m(\angle DCA) + m(\angle ACB) = m(\angle DCB)$. **(Justify.)**
(6) $m(\angle DCB) + m(\angle BCE) = 180$. **(Justify.)**
(7) $m(\angle DCA) + m(\angle ACB) + m(\angle BCE) = 180$. **(Justify.)**

(8) $m(\angle DCA) = m(\angle CAB)$. **(Justify.)**
(9) $m(\angle BCE) = m(\angle CBA)$. **(Justify.)**
(10) $m(\angle CAB) + m(\angle ACB) + m(\angle CBA) = 180$. **(Justify.)**
End of Proof

There are two straightforward corollaries of Theorem 103. The first has to do with exterior angles. Recall the statement of the Neutral Exterior Angle Theorem, presented in Section 7.3:

Theorem 59: Neutral Exterior Angle Theorem
In Neutral Geometry, the measure of any exterior angle is greater than the measure of either of its remote interior angles.

You may have a vague memory of something called the Exterior Angle Theorem from your high school geometry class, and you might remember that its statement did not resemember the statement above. The reason is that in high school, you learned a statement about exterior angles that is true in Euclidean Geometry, but that is not true in Neutral Geometry. here is the statement, presented as a theorem of Euclidean Geometry:

Theorem 104 (corollary) Euclidean Exterior Angle Theorem.
In Euclidean Geometry, the measure of any exterior angle is equal to the sum of the measure of its remote interior angles
Proof:
(1) In Euclidean Geometry, suppose that a triangle and an exterior angle for it are given.
(2) Label the vertices of the triangle A, B, C such that the given exterior angle is $\angle CBD$, with $A * B * D$.
is given. **(make a drawing)**
(3) $m(\angle ABC) + m(\angle CBD) = 180$. **(Justify.)**
(4) $m(\angle ABC) + m(\angle BCA) + m(\angle CAB) = 180$. **(Justify.)**
(5) $m(\angle CBD) = m(\angle BCA) + m(\angle CAB)$. **(Justify.)**
End of Proof

You will justify the above proof in a homework exercise.

It is important to keep in mind that we now have two theorems about Exterior Angles.
- **Theorem 59** is called the Neutral Exterior Angle Theorem; it is a valid theorem in Neutral Geometry and also in Euclidean Geometry.
- **Theorem 104** is called the Euclidean Exterior Angle Theorem; it is a valid theorem in Euclidean Geometry but not in Neutral Geometry.

Because there are two theorems about Exterior Angles, it is imperative that you use the full name of the theorems when referring to them. In particular, it is not acceptable to justify a step in a proof by citing "the Exterior Angle Theorem". You must indicate which one.

The second corollary of Theorem 103 that we will study deals with angle sum of convex quadrilaterals. You will prove it in a homework exercise.

Theorem 105 (corollary) In Euclidean Geometry, the angle sum of any convex quadrilateral is 360.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.4. In Euclidean Geometry, every triangle can be circumscribed

In Section 8.4 we proved Theorem 92 (in Neutral Geometry, the three angle bisectors of any triangle are concurrent at a point that is equidistant from the three sides of the triangle.) (found on page 202). The point of concurrence is called the *incenter* of the triangle. Using that concurrence theorem, we were able in Section 8.5 to prove Theorem 95 (in Neutral Geometry, every triangle has exactly one inscribed circle.) (found on page 204). We saw that the inscribed circle has its center at the *incenter* of the triangle. That's why the name *incenter* is used for that point. All that stuff remains valid in Euclidean Geometry, of course.

Now we will study a different concurrence statement, this one about the perpendicular bisectors of the sides of a triangle. We will prove that in Euclidean Geometry, the three perpendicular bisectors are concurrent. And we will be able to use that concurrence theorem to help us prove the existence of another special circle related to the triangle. But the things that we prove here are only valid in Euclidean Geometry. They are not valid statements in Neutral Geometry.

Here is the concurrence theorem. You will justify it in a class drill.

Theorem 106 In Euclidean Geometry, the perpendicular bisectors of the three sides of any triangle are concurrent at a point that is equidistant from the vertices of the triangle. (This point will be called the circumcenter.)

Proof

(1) Suppose that ΔABC is given in Euclidean Geometry.

(2) There exists a line L that is the perpendicular bisector of side $\overline{AB}$. **(Justify.)** Let D be the point of intersection of line L and side $\overline{AB}$.

(3) There exists a line M that is the perpendicular bisector of side $\overline{BC}$. **(Justify.)** Let E be the point of intersection of line M and side $\overline{BC}$.

(4) There exists a line N that is the perpendicular bisector of side $\overline{CA}$. **(Justify.)** Let F be the point of intersection of line N and side $\overline{CA}$.

(5) We must show that there exists a point that all three lines L, M, N pass through.)

Show that lines L and M intersect. (indirect proof)

(6) Assume that lines L and M do not intersect. **(Justify.)** Then lines L and M are parallel.

(7) Observe that line $\overleftrightarrow{BC}$ is perpendicular to line M.

(8) Therefore, line $\overleftrightarrow{BC}$ is also perpendicular to line L. **(Justify.)**

(9) Observe that line $\overleftrightarrow{AB}$ is perpendicular to line L.

(10) Line $\overleftrightarrow{AB}$ is not the same line as line $\overleftrightarrow{BC}$. **(Justify.)**

(11) So there are two lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$, that pass through point B and are perpendicular to line L.

(12) Statement (11) contradicts something. (What does it contradict?) Therefore, our assumption in step (6) was wrong. Lines L and M must intersect. Let G be their point of intersection.

Prove that point G is equidistant from the vertices of the triangle and is on all three perpendicular bisectors.

(13) Observe that G is on line L, which is the perpendicular bisector of side $\overline{AB}$.
(14) $GA = GB$. **(Justify.)**
(15) Observe that G is on line M, which is the perpendicular bisector of side $\overline{BC}$.
(16) $GB = GC$. **(Justify.)**
(17) $GA = GC$. **(Justify.)**
(18) G is on line N, which is the perpendicular bisector of side $\overline{CA}$. **(Justify.)**

End of proof

The point where the perpendicular bisectors of the three sides intersect is called the circumcenter of the triangle.

Definition 71 circumcenter of a triangle
The circumcenter of a triangle in Euclidean Geometry is defined to be the point where the perpendicular bisectors of the three sides intersect. (Such a point is guaranteed to exist by Theorem 106)

The name *circumcenter* might sound strange, but the reason for the terminology will become clearer after we present the following definition and theorem.

Definition 72 a circle circumscribes a triangle
We say that a circle *circumscribes* a triangle if the circle passes through all three vertices of the triangle.

Theorem 107 (corollary) In Euclidean Geometry, every triangle can be circumscribed.

Proof

(1) Suppose that ΔABC is given in Euclidean Geometry.
(2) There exists a point G such that $GA = GB = GC$. (by Theorem 106)
(3) Define the real number $r = GA = GB = GC$. Observe that $Circle(G,r)$ passes through all three vertices A, B, C.

End of proof

We see that the circle that circumscribes a triangle has its center at the point where the perpendicular bisectors meet. This point was given the name *circumcenter* in Definition 71. Now we see why the name circumcenter is used: the circumcenter is the center of the circle that circumscribes the circle.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.5. Parallelograms in Euclidean Geometry

In this section, we turn our attention to parallelograms. Here is the definition.

Definition 73 parallelogram
A parallelogram is a quadrilateral with the property that both pairs of opposite sides are parallel.

Notice that there is nothing in the definition of parallelogram that assumes Euclidean Geometry. Indeed, parallelograms do exist in Neutral Geometry. But there are not general properties that are true for all parallelograms in Neutral Geometry. However, because of the additional restrictions placed on parallel lines in Euclidean Geometry, it turns out that that there are general properties that can be identified for parallelograms in Neutral Geometry. The following theorem tells us that there are a bunch of statements about convex quadrilaterals in Euclidean geometry that all turn out to be equivalent to saying that the convex quadrilateral is a parallelogram.

Theorem 108 equivalent statements about convex quadrilaterals in Euclidean Geometry
In Euclidean Geometry, given any convex quadrilateral, the following statements are equivalent (TFAE)
(i) Both pairs of opposite sides are parallel. That is, the quadrilateral is a parallelogram.
(ii) Both pairs of opposite sides are congruent.
(iii) One pair of opposite sides is both congruent and parallel.
(iv) Each pair of opposite angles is congruent.
(v) Either diagonal creates two congruent triangles.
(vi) The diagonals bisect each other.

The proof is left as an exercise

The theorem just presented is an equivalence theorem. The last time we encountered an equivalence theorem with so many statements was back in Section 7.7, when we studied Theorem 73 (Equivalent statements about angles formed by two lines and a transversal in Neutral Geometry) (found on page 186). Following that theorem were two remarks:

- A Remark about Proving an Equivalence Theorem
- A Remark about "Using" an Equivalence Theorem

It would be useful for the reader to take a moment to review those remarks.

There is an immediate corollary that you will be asked to prove in a homework exercise. Note that the theorem statement does not mention parallelograms. But you should use them in your proof.

Theorem 109 (corollary) In Euclidean Geometry, parallel lines are everywhere equidistant.
In Euclidean Geometry, if lines K and L are parallel, and line M is a transversal that is perpendicular to lines K and L at points A and B, and line N is a transversal that is perpendicular to lines K and L at points C and D, then $AB = CD$.

Parallel lines can be used in the proof of the concurrence of special lines associated to triangles. In the next section, we will prove that for any triangle in Euclidean Geometry, the three altitude lines are concurrent. In the final section of the chapter, we will prove that for any triangle in Euclidean Geometry, the three medians are concurrent.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.6. The triangle midsegment theorem and altitude concurrence

In this section, we will prove that for any triangle in Euclidean Geometry, the three altitude lines are concurrent. The theorem can be proven directly, with a rather hard proof. But I find it more interesting to first introduce midsegments and medial triangles and prove some properties of those. Then, the proof of altitude concurrence will turn out to be an easy corollary. Here is the definition of midsegment.

Definition 74 midsegment of a triangle

A midsegment of a triangle is a line segment that has endpoints at the midpoints of two of the sides of the triangle.

As far as I can remember, triangle midsegments only appear in the statement of one theorem, and it is a very simple-sounding theorem. But it turns out to be a very useful theorem. Here it is:

Theorem 110 The Euclidean Geometry Triangle Midsegment Theorem

In Euclidean Geometry, if the endpoints of a line segment are the midpoints of two sides of a triangle, then the line segment is parallel to the third side and is half as long as the third side. That is, a midsegment of a triangle is parallel to the third side and half as long.

Proof

(1) In Euclidean Geometry, suppose that the endpoints of a line segment are the midpoints of two sides of a triangle. Label the triangle ΔABC so that the line segment is $\overline{DE}$, with endpoints D and E being the midpoints of sides $\overline{AB}$ and $\overline{AC}$. **(Make a drawing.)**

(2) There exists a point F such that $D * E * F$ and $\overline{EF} \cong \overline{ED}$. **(Justify.) (Make a new drawing.)**

(3) $\Delta CEF \cong \Delta AED$. **(Justify.) (Make a new drawing.)**

(4) $\angle FCE \cong \angle DAE$ and $\overline{CF} \cong \overline{AD}$. **(Justify.) (Make a new drawing.)**

(5) $\overline{CF} \cong \overline{BD}$. **(Justify.) (Make a new drawing.)**

(6) Line $\overleftrightarrow{CF}$ is parallel to line $\overleftrightarrow{AD}$. **(Justify.) (Make a new drawing.)**

(7) Quadrilateral $\square BCFD$ is a parallelogram. **(Justify.)**

(8) Segment $\overline{DF} \cong \overline{BC}$ and line $\overleftrightarrow{DF}$ is parallel to line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(9) Midsegment $\overline{DE}$ is parallel to side $\overline{BC}$. **(Justify.) (Make a new drawing.)**

(10) $DE = \frac{1}{2}DF$. **(Justify.)**

(11) $DE = \frac{1}{2}BC$. **(Justify.)**

End of proof

Now for the definition of medial triangles.

Definition 75 medial triangle

Words: Triangle #1 is the medial triangle of triangle #2.

Meaning: The vertices of triangle #1 are the midpoints of the sides of triangle #2.

Additional Terminology: We will refer to triangle #2 as the outer triangle.

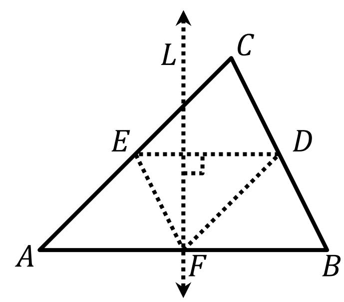
The Euclidean GeometryTriangle Midsegment Theorem that we just proved will enable us to easily prove some properties of Medial Triangles in Euclidean Geometry. Here is a theorem stating the properties.

Theorem 111 Properties of Medial Triangles in Euclidean Geometry

(1) The sides of the medial triangle are parallel to sides of outer triangle and are half as long.
(2) The altitude lines of the medial triangle are the perpendicular bisectors of the sides of the outer triangle.
(3) The altitude lines of the medial triangle are concurrent.

Proof

Let triangle ΔDEF be the medial triangle of outer triangle ΔABC, labeled so that vertices D, E, F are the midpoints of the sides opposite vertices A, B, C.



Proof of (1)

Observe that each side of medial triangle ΔDEF is a midsegment of outer triangle ΔABC. Therefore, the sides of triangle ΔDEF will be parallel to the sides of outer triangle ΔABC and half as long. **(Justify.)**

Proof of (2)

Consider the altitude line L that passes through vertex F of medial triangle ΔDEF. This line is perpendicular to line $\overleftrightarrow{DE}$. But line $\overleftrightarrow{DE}$ is parallel to line $\overleftrightarrow{AB}$.
Therefore, line L is also perpendicular to line $\overleftrightarrow{AB}$. **(Justify.)** Since line L passes through point F that is the midpoint of segment $\overline{AB}$, we see that line L is the perpendicular bisector of side $\overline{AB}$. Similar reasoning applies to the other two altitudes of medial triangle ΔDEF.

Proof of (3)

The perpendicular bisectors of the three sides of outer triangle ΔABC are concurrent.**(Justify)** But those three perpendicular bisectors are the altitude lines of medial triangle ΔDEF.
Therefore, the altitude lines of medial triangle ΔDEF are concurrent.

End of proof

Remember that our goal for this section was to prove that for any given triangle, the three altitude lines are concurrent. So far, we have only proved that the three altitude lines of a medial triangle are concurrent. This next theorem is the final piece that we need to complete the puzzle.

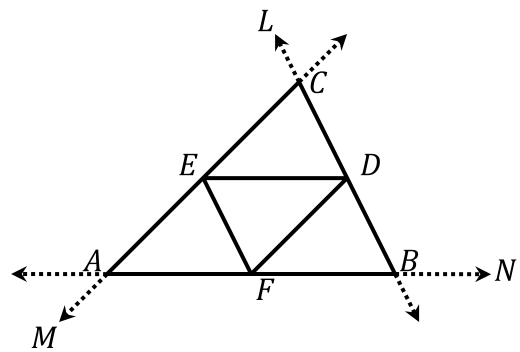
Theorem 112 In Euclidean Geometry any given triangle is a medial triangle for some other.

Proof

(1) Suppose that triangle ΔDEF is given in Neutral Geometry.

Introduce lines $\boldsymbol{L, M, N}$.

(2) Let line L be the unique line that passes through vertex D and is parallel to line $\overleftrightarrow{EF}$.
(3) Let line M be the unique line that passes through vertex E and is parallel to line $\overleftrightarrow{FD}$.
(4) Let line N be the unique line that passes through vertex F and is parallel to line $\overleftrightarrow{DE}$.



Introduce triangle ΔABC.

(5) Lines L and M intersect at a point that we can call C. (If lines L and M did not intersect, then they would be parallel, and Theorem 99 tell that that lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{FD}$ would be parallel as well. But we know that lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{FD}$ are not parallel because they intersect at point F.)

(6) Similarly, lines M and N intersect at a point that we can call A, and lines N and L intersect at a point that we can call B.

(7) Clearly, points A, B, C are non-collinear. (If they were collinear, then all three lines L, M, N would be the same line. That would mean that the three lines $\overleftrightarrow{DE}, \overleftrightarrow{EF}, \overleftrightarrow{FD}$ are parallel. But they are not.) So there is a triangle ΔABC.

Use parallelograms to prove that points D, E, F are midpoints.

(8) Observe that Quadrilateral $\square EFDC$ is a parallogram.

(9) Therefore, $\overline{EF} \cong \overline{DC}$. (by Theorem 108)

(10) And observe that Quadrilateral $\square EFBD$ is a parallogram.

(11) Therefore, $\overline{EF} \cong \overline{BD}$. (by Theorem 108)

(12) Therefore, point D is the midpoint of side $\overline{BC}$.

(13) Similarly, point E is the midpoint of side $\overline{CA}$ and point F is the midpoint of side $\overline{AB}$.

Conclusion

(14) Triangle ΔDEF is the medial triangle for triangle ΔABC.

End of Proof

We are now able to easily prove that the altitude lines of any triangle in Euclidean Geometry are concurrent. We will call the point of concurrence the *orthocenter*.

Theorem 113 (Corollary) In Euclidean Geometry, the altitude lines of any triangle are concurrent.

Proof

(1) Suppose that a triangle is given in Euclidean Geometry.

(2) That triangle is a medial triangle. **(Justify.)**

(3) Therefore the three altitude lines are concurrent. **(Justify.)**

End of proof

Definition 76 Orthocenter of a triangle in Euclidean Geometry

The orthocenter of a triangle in Euclidean Geometry is a point where the three altitude lines intersect. (The existence of such a point is guaranteed by Theorem 113.)

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 9.8 on page 222.

9.7. Advanced Topic: Equally-spaced parallel lines and median concurrence

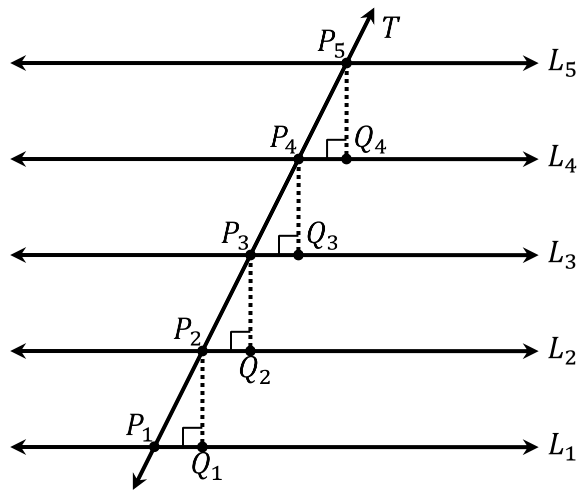
Recall that the median of a triangle was introduced in Definition 66. Our goal is a theorem stating that in Euclidean Geometry, the three medians of any triangle are concurrent. Our proof will involve equally-spaced lines. Recall that Theorem 109 tells us that in Euclidean Geometry, parallel lines are everywhere equidistant. That theorem is referring to a single pair of parallel lines. The term *equally-spaced lines* refers to a collection of lines.

Definition 77 Equally-Spaced Parallel Lines in Euclidean Geometry
Words: lines $L_1, L_2, \cdots, L_n$ are equally-spaced parallel lines.
Meaning: The lines are parallel and $L_1L_2 = L_2L_3 = \cdots = L_{n-1}L_n$.

Equally-spaced lines will be important to us because they cut congruent segments in transversals. The following theorem makes this precise.

Theorem 114 about n distinct parallel lines intersecting a transversal in Euclidean Geometry
Given: in Euclidean Geometry, parallel lines $L_1, L_2, \cdots, L_n$ intersecting a transversal T at points $P_1, P_2, \cdots, P_n$ such that $P_1 * P_2 * \cdots * P_n$.
Claim: The following are equivalent
(i) Lines $L_1, L_2, \cdots, L_n$ are equally spaced parallel lines.
(ii) The lines cut congruent segments in transversal T. That is, $\overline{P_1P_2} \cong \overline{P_2P_3} \cong \cdots \cong \overline{P_{n-1}P_n}$.

A picture is shown at right. In a homework exercise, you will use this picture as an illustration for a proof of Theorem 114 in the case of five lines L_1, L_2, L_3, L_4, L_5. The general statement involving n lines $L_1, L_2, \cdots, L_n$ is proven using induction. We won't do that more general proof in this course.

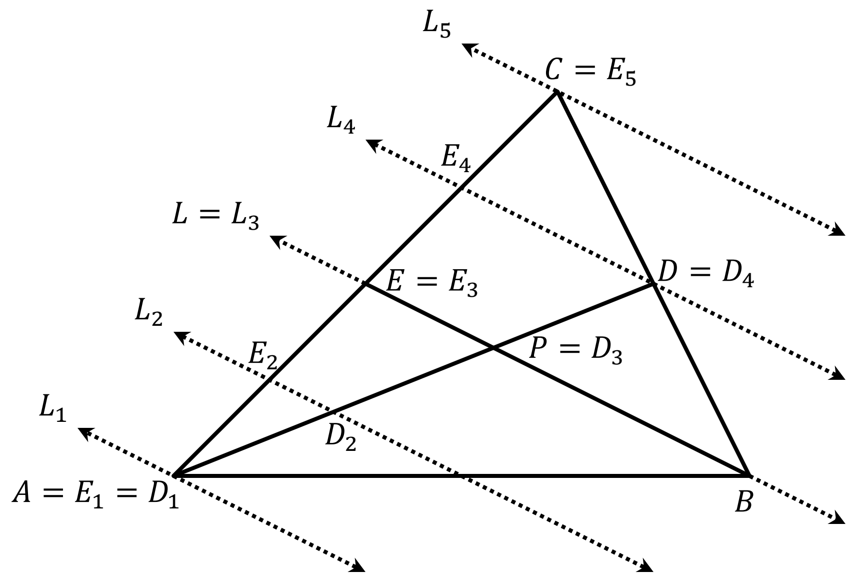


The following easy corollary requires no proof.

Theorem 115 (Corollary) about n distinct parallel lines cutting congruent segments in transversals in Euclidean Geometry
If a collection of n parallel lines cuts congruent segments in one transversal, then the n parallel lines must be equally spaced and so they will also cut congruent segments in any transversal.

We are now ready to prove the concurrence of medians of triangles in Euclidean Geometry.

Theorem 116 about concurrence of medians of triangles in Euclidean Geometry
In Euclidean Geometry, the medians of any triangle are concurrent at a point that can be called the centroid. Furthermore, the distance from the centroid to any vertex is $2/3$ the length of the median drawn from that vertex.



Proof

- Let D, E, F be the midpoints of side $\overline{BC}, \overline{CA}, \overline{AB}$.

Part 1: Prove that medians $\overline{AD}$ and $\overline{BE}$ intersect at a point P in the interior of ΔABC.

- You do this.

Part 2: Prove that $AP = \frac{2}{3}AD$.

- Let E_2 be the midpoint of $\overline{AE}$, and let E_4 be the midpoint of $\overline{CE}$.
- It is convenient to use the alternate names $A = E_1$ and $C = E_5$ so that we can observe that the five points E_1, E_2, E_3, E_4, E_5 are equally spaced on line $\overleftrightarrow{AC}$.
- Let L be median line $\overleftrightarrow{BE}$.
- Let L_1, L_2, L_4, L_5 be the unique lines that pass through points E_1, E_2, E_4, E_5 and are parallel to L. **(justify)**
- It is convenient to use the alternate name $L = L_3$ so that we can observe that the five lines L_1, L_2, L_3, L_4, L_5 cut congruent segments in line $\overleftrightarrow{AC}$.
- Therefore, the three lines L_3, L_4, L_5 also cut congruent segments in line $\overleftrightarrow{BC}$. **(justify)** This tells us that line L_4 must pass through midpoint D of segment $\overline{BC}$.
- And, the five lines L_1, L_2, L_3, L_4, L_5 also cut congruent segments in line $\overleftrightarrow{AD}$. **(justify)** That is, if $D_1, D_2, D_3, D_4 = D$ are the points of intersection of lines L_1, L_2, L_3, L_4 and line $\overleftrightarrow{AD}$, then $\overline{D_1D_2} \cong \overline{D_2D_3} \cong \overline{D_3D_4}$.
- Conclude that $AP = \frac{2}{3}AD$.

Part 3: Prove that $BP = \frac{2}{3}BE$.

- Repeat the process of Part 2, but this time use five points $D_1 = B, D_2, D_3 = D, D_4, D_5 = C$ that are equally spaced on line $\overleftrightarrow{BC}$.
- Let L_1, L_2, L_4, L_5 be the unique lines that pass through points D_1, D_2, D_4, D_5 and are parallel to median line $\overleftrightarrow{AD} = L = L_3$.

- Let $E_1 = B, E_2, E_3, E_4 = E$ are the points of intersection of lines L_1, L_2, L_3, L_4 and line $\overleftrightarrow{BE}$.
- Show that $\overline{E_1E_2} \cong \overline{E_2E_3} \cong \overline{E_3E_4}$.
- Conclude that $BP = \frac{2}{3}BE$.

Part 4: Prove that medians $\overline{BE}$ and $\overline{CF}$ intersect at a point Q in the interior of ΔABC.
- You do this.

Part 5: Prove that $CQ = \frac{2}{3}CF$.
- You do this.

Part 6: Prove that $BQ = \frac{2}{3}BE$.
- You do this.

Conclusion
- Conclude that points P and Q must be the same point.

End of proof

The term centroid was introduced in the above theorem. Here is the official defintion.

Definition 78 Centroid of a triangle in Euclidean Geometry
The centroid of a triangle in Euclidean Geometry is the point where the three medians intersect. (Such a point is guaranteed to exist by Theorem 116.)

It turns out that also in Neutral Geometry, not just in Euclidean Geometry, the three medians of any triangle are concurrent. So any triangle in Neutral Geometry has a centroid. You might wonder why we did not prove that more general fact back when we were studying Neutral Geometry. The proof that we just did is a Euclidean proof: it uses concepts of equally-spaced lines and unique parallels, things that only happen in Euclidean Geometry. There is another proof that works only in Hyperbolic geometry.. So taken together, those constitute a proof of median concurrence in Neutral Geometry. I know of no proof of median concurrence in Neutral Geometry that does not use two cases, one for the Euclidean case and one for the Hyperbolic case. So I know of no proof that would have been appropriate for our earlier chapters on Neutral Geometry.

9.8. Exercises for Chapter 9

Exercises for Section 9.1 Introduction (Section starts on page 209)

[1] Justify the steps in the proof of Theorem 97 ((Corollary) In Euclidean Geometry, the answer to the recurring question is *exactly one line*.) (found on page 210).

[2] Justify the steps in the proof of Theorem 98 ((corollary) In Euclidean Geometry, if a line intersects one of two parallel lines, then it also intersects the other.) (found on page 210).

[3] Justify the steps in the proof of Theorem 99 ((corollary) In Euclidean Geometry, if two distinct lines are both parallel to a third line, then the two lines are parallel to each other.) (found on page 211).

Exercises for Section 9.2 Parallel Lines and Alternate Interior Angles in Euclidean Geometry (Section starts on page 211)

[4] Prove Theorem 102 ((corollary) In Euclidean Geometry, if a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other. That is, if lines L and M are parallel, and line T is perpendicular to M, then T is also perpendicular to L.) (found on page 212). (Hint: Remember the proof structure: The given information goes in step (1). Then show that T intersects L. Then show that T is perpendicular to L. Justify all steps.

Exercises for Section 9.3 Angles of Triangles in Euclidean Geometry (Starts on page 212)

[5] Justify the steps in the proof of Theorem 103 (In Euclidean Geometry, the angle sum for any triangle is 180.) (found on page 212).

[6] Justify the steps in the proof of Theorem 104 ((corollary) Euclidean Exterior Angle Theorem.) (found on page 213).

[7] Prove Theorem 105 ((corollary) In Euclidean Geometry, the angle sum of any convex quadrilateral is 360.) (found on page 213). Hint: The convex quadrilateral has four angles. Draw a diagonal to create two triangles, thus six angles. Use what you know about triangle angle sums to determine the sum of the measures of the six angles. You would like to be able to say that that sum will be the same as the sum of the four angles of the quadrilateral, but that will require angle addition. Before using angle addition, you will have to show that certain requirements are met.

Exercises for Section 9.4 In Euclidean Geometry, every triangle can be circumscribed (Section starts on page 214)

[8] Justify the steps in the proof of Theorem 106 (In Euclidean Geometry, the perpendicular bisectors of the three sides of any triangle are concurrent at a point that is equidistant from the vertices of the triangle. (This point will be called the circumcenter.)) (found on page 214).

Exercises for Section 9.5 Parallelograms in Euclidean Geometry (Starts on page 215)

[9] Prove Theorem 108 (equivalent statements about convex quadrilaterals in Euclidean Geometry) (found on page 216).
Hint: Be sure to re-read the "Remark about Proving an Equivalence Theorem" that followed the presentation of Theorem 73 (Equivalent statements about angles formed by two lines and a transversal in Neutral Geometry) on page 186 of Section 7.7.

[10] Prove Theorem 109 ((corollary) In Euclidean Geometry, parallel lines are everywhere equidistant.) (found on page 216).

Exercises for Section 9.6 The triangle midsegment theorem and altitude concurrence (Section starts on page 217)

[11] Justify the steps in the proof of Theorem 110 (The Euclidean Geometry Triangle Midsegment Theorem) (found on page 217).

[12] Prove that given any convex quadrilateral $ABCD$, if the midpoints E, F, G, H of the four sides are joined to form a new quadrilateral $EFGH$, then $EFGH$ is a parallelogram.

[13] Justify the steps in the proof of Theorem 111 (Properties of Medial Triangles in Euclidean Geometry) (found on page 218).

[14] Justify the steps in the proof of Theorem 113 ((Corollary) In Euclidean Geometry, the altitude lines of any triangle are concurrent.) (found on page 219).

Exercises for Section 9.7 (Advanced Topic: Equally-spaced parallel lines and median concurrence) (Section starts on page 220)

[15] (Advanced) Prove Theorem 114 (about n distinct parallel lines intersecting a transversal in Euclidean Geometry) (found on page 220).

[16] (Advanced) Justify the steps and supply the missing steps in Theorem 116 (about concurrence of medians of triangles in Euclidean Geometry) (found on page 220).

10. Euclidean Geometry II: Similarity

Triangle Congruence was introduced in Definition 54 of Chapter 7. That definition did not tell us anything about the behavior of triangle congruence. It was the axiom systems for Neutral Geometry (Definition 17) and Euclidean Geometry (Definition 70) that gave us information about how triangle congruence behaved. Those axiom systems included an axiom about triangle congruence (The Side-Angle-Side Axiom, <N10>). In Chapter 7, we studied theorems that involved triangle congruence. The theorems in Chapter 7 were theorems of Neutral Geometry. That means that those theorems are true in both Neutral Geometry and Euclidean Geometry. The theorems in Chapters 9 were theorems of Euclidean Geometry: their proofs also depended on the Euclidean Parallel Axiom, <EPA>. Keep in mind that without Axiom <N10>, we would not know anything about the behavior of triangle congruence and therefore we would not be able to prove many of the theorems of Chapters 7 through 9.

In this chapter, we will define the concept of *similarity* in Euclidean Geometry and we will use that concept to prove some theorems, including the famous Pythagorean Theorem. There are no axioms that mention *similarity*. All that we will know about *similarity* will be a consequence of the eleven axioms that we already have discussed: Axioms <N1> through <N10> and <EPA>. Any theorems that we later prove using the concept of *similarity* would be true statements in Euclidean Geometry even if we did not introduce the concept of *similarity*. It is reasonable to wonder why we bother introducing *similarity* if it does not make anything true that would not be true without introducing *similarity*. The reason is that the concept of *similarity* allows us to shorten proofs. That is, it may take 10 or 20 pages of this text to develop the ideas of *similarity*, but then the proof of the Pythagorean Theorem is only about 10 lines long. And a bunch of other theorems are also made short. If we did not develop the ideas of *similarity*, then the proof of the Pythagorean Theorem and many other theorems would each need to be 10 or 20 pages long. Because the concept of *similarity* can shorten so many proofs, it is worth developing.

Our first theorems about *similarity* will be proven using properties of parallel lines. In particular, we will use something called *parallel projection*. The most basic properties of *parallel projection* have simple proofs. So we will start this chapter with a section on *parallel projection* and then get to *similarity* in the section after that.

10.1. Parallel Projections

In most of this book, we have studied the objects of geometry. We have had theorems about points, lines, angles, triangles, circles, and quadrilaterals. *Parallel projection* is interesting because it is easiest to *explain* parallel projection using drawings of points and lines, and yet it is useful to *define* parallel projection using the terminology of functions. Here is the definition.

Definition 79 Parallel Projection in Euclidean Geometry

Symbol: $Proj_{L,M,T}$

Usage: L, M, T are lines, and T intersects both L and M.

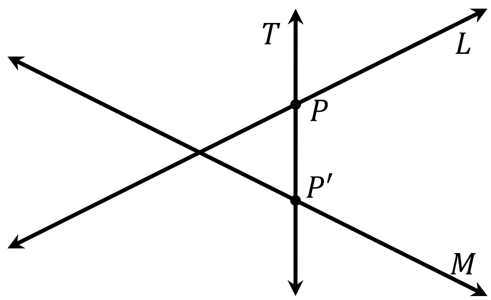
Meaning: $Proj_{L,M,T}$ is a function whose domain is the set of points on line L and whose codomain is the set of points on line M. In function notation, this would be denoted by the symbol $Proj_{L,M,T}: L \rightarrow M$. Given an input point P on line L, the output point on line

M is denoted P'. That is, $P' = Proj_{L,M,T}(P)$. The output point P' is determined in the following way:

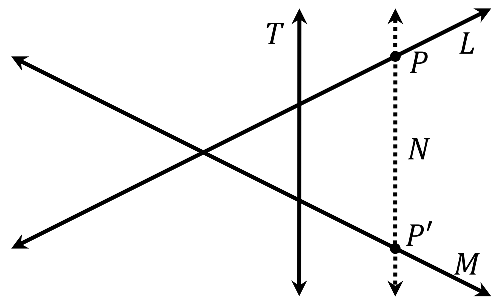
Case 1: If P happens to lie at the intersection of lines L and T, then P' is defined to be the point at the intersection of lines M and T.

Case 2: If P lies on L but not on T, then there exists exactly one line N that passes through P and is parallel to line T. (Such a line N is guaranteed by Theorem 97). The output point P' is defined to be the point at the intersection of lines M and N.

Drawing:



Case 1: P lies on both L and T.



Case 2: P lies on L but not on T.

Our first three theorems about parallel projection have fairly straightforward proofs. We will prove all three theorems.

Theorem 117 Parallel Projection in Euclidean Geometry is *one-to-one* and *onto*.

A digression to discuss *one-to-one* and *onto*.

Before proving this theorem, it is useful to digress and review what it means to say that a function is *one-to-one* and *onto*, and to discuss strategies for proving that a function has those properties.

To say that a function $f: A \to B$ is *one-to-one* means that different inputs always result in different outputs. That is, $\forall x_1, x_2 \in A, if\ x_1 \neq x_2\ then\ f(x_1) \neq f(x_2)$. The contrapositive of this statement has the same meaning. It says $\forall x_1, x_2 \in A, if\ f(x_1) = f(x_2)\ then\ x_1 = x_2$. That is, if the two outputs are the same, then the two inputs must have been the same.

To say that a function $f: A \to B$ is *onto* means that for any chosen element of the codomain, there exists some element of the domain that can be used as input and will produce the chosen element of the codomain as output. That is, $\forall y \in B, \exists x \in A\ such\ that\ f(x) = y$.

It can be difficult to prove that a function is *one-to-one* or *onto*. Remember that if a function $f: A \to B$ is *one-to-one* and *onto*, then it has an inverse function, denoted $f^{-1}: B \to A$. An inverse function is a function that satisfies both of the following inverse relations:

$$\forall x \in A, f^{-1}(f(x)) = x$$
$$\forall y \in B, f(f^{-1}(y)) = y$$

For example, the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ has an inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = x^{\frac{1}{3}}$. To verify that this is indeed the inverse function, we have to check to see if the inverse relations are satisfied.

$$f^{-1}(f(x)) = ((x)^3)^{\frac{1}{3}} = x$$
$$f(f^{-1}(x)) = \left((x)^{\frac{1}{3}}\right)^3 = x$$

Since both inverse relations are satisfied, we have confirmed that $f^{-1}(x) = x^{\frac{1}{3}}$ is indeed the inverse function for $f(x) = x^3$.

Now also remember that if it is known that a function f has an inverse function, then f must be both *one-to-one* and *onto*. That is, given some $f: A \to B$, if one can somehow find a function $g: B \to A$ that satisfies the two equations

$$\forall x \in A, g(f(x)) = x$$
$$\forall y \in B, f(g(y)) = y$$

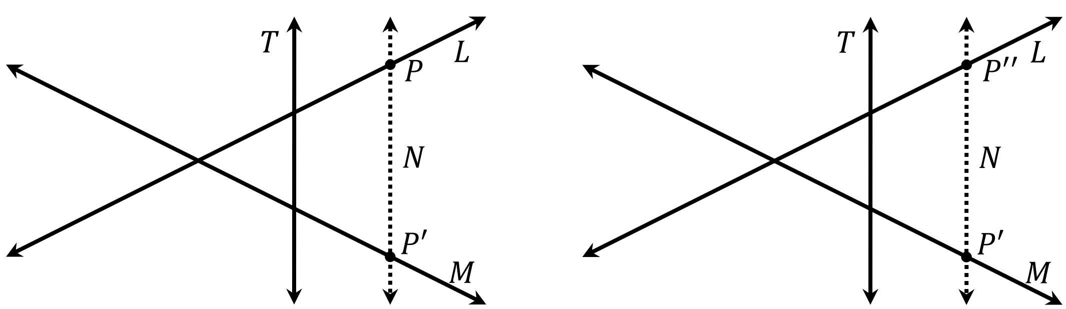
then g is the inverse function for f and one would automatically know that f must be both *one-to-one* and *onto*.

End of the digression

In light of what was discussed in the digression about *one-to-one* and *onto*, we see that one strategy for proving that the projection $Proj_{L,M,T}: L \to M$ is *one-to-one* and *onto* would be to somehow find a function $g: M \to L$ that qualifies as an inverse function. That is the strategy that we will take.

Proof of Theorem 117.

Given the parallel projection $Proj_{L,M,T}: L \to M$, consider the projection $Proj_{M,L,T}: M \to L$. This is a function that takes as input a point on line M and produces as output a point on line L. For any point P on line L, find the value of $Proj_{M,L,T}\left(Proj_{L,M,T}(P)\right)$.



Referring to the sample diagram above, we see that

$$Proj_{M,L,T}\left(Proj_{L,M,T}(P)\right) = Proj_{M,L,T}(P') = P''$$

But notice that the output point P'' is the same as the input point P. That is,

$$Proj_{M,L,T}\left(Proj_{L,M,T}(P)\right) = P$$

A similar drawing would show that for any point Q on line M,

$$Proj_{L,M,T}\left(Proj_{M,L,T}(Q)\right) = Q$$

Therefore the projection $Proj_{M,L,T}: M \rightarrow L$ is qualified to be called the inverse function for the projection $Proj_{L,M,T}: L \rightarrow M$. Since the projection $Proj_{L,M,T}: L \rightarrow M$ has an inverse function, we conclude that it is both *one-to-one* and *onto*.

End of proof

Our second basic theorem about parallel projection can be proven using concepts from Chapter 3. Recall that betweenness of points was introduced in Definition 24. The following theorem articulates what happens when three points with a particular betweenness relationship are used as input to a Parallel Projection function. The theorem says that the resulting three output points have the same betweenness relationship.

Theorem 118 Parallel Projection in Euclidean Geometry preserves betweenness.
If L, M, T are lines, and T intersects both L and M, and A, B, C are points on L with $A * B * C$, then $A' * B' * C'$.

Proof (for readers interested in advanced topics and for graduate students)
The proof is left to an exercise.

Our third basic theorem about parallel projection is proved using facts about parallelograms—facts that we studied in Section 9.5. In a homework exercise, you will be asked to make a drawing to illustrate the proof.

Theorem 119 Parallel Projection in Euclidean Geometry preserves congruence of segments.
If L, M, T are lines, and T intersects both L and M, and A, B, C, D are points on L with $\overline{AB} \cong \overline{CD}$, then $\overline{A'B'} \cong \overline{C'D'}$.

Proof

(1) Suppose that L, M, T are lines, and T intersects both L and M, and A, B, C, D are points on L with $\overline{AB} \cong \overline{CD}$. **(Make a drawing.)**

Part 1 Introduce lines and points.

(2) Let N_1, N_2, N_3, N_4 be lines that pass through A, B, C, D and are parallel to line T. (One of these lines could actually be line T.) **(Update your drawing.)**

(3) Let $A' = Proj_{L,M,T}(A)$, and similarly for B', C', D'. Then the four points A', B', C', D' are located at the intersections of lines N_1, N_2, N_3, N_4 and line M. **(Update your drawing.)**

(4) Let K_1 be the line that passes through point A and is parallel to line M. **(Update your drawing.)**

(5) Let K_2 be the line that passes through point C and is parallel to line M. **(Update your drawing.)**

(6) Let E be the point at the intersection of lines K_1 and N_2. **(Update your drawing.)**

(7) Let F be the point at the intersection of lines K_2 and N_4. **(Update your drawing.)**

Part 2: Show that two triangles are congruent.

(8) Observe that $\angle ABE \cong \angle CDF$ (by Theorem 100 applied to parallel lines K_1, K_2 and transversal L).

(9) Observe that $\angle BAE \cong \angle DCF$. **(Justify.)**

(10) Therefore, that $\Delta ABE \cong \Delta CDF$. **(Justify.)**

Part 3: Prove that the segments are congruent.

(11) $\overline{AE} \cong \overline{CF}$ (by statement (10) and the definition of triangle congruence, Definition 54)

(12) Observe that $Quadrilateral(AEB'A')$ is a parallelogram.

(13) Therefore, $\overline{AE} \cong \overline{A'B'}$. **(Justify.)**

(14) Observe that $Quadrilateral(CFD'C')$ is a parallelogram.

(15) Therefore, $\overline{CF} \cong \overline{C'D'}$. **(Justify.)**

(16) Conclude that $\overline{A'B'} \cong \overline{C'D'}$ (by statements (13), (11), (15), and transitivity).

End of Proof

Our fourth theorem about parallel projection can be proved using concepts that are at the level of this book, but it would take a day of lecture time and a couple of additional sections of text to fully present. All that would be okay, but the proof is of a style that would not appear again in the course. For that reason, the proof is left as an advanced exercise.

Theorem 120 Parallel Projection in Euclidean Geometry preserves ratios of lengths of segments. If L, M, T are lines, and T intersects both L and M, and A, B, C, D are points on L with $C \neq D$, then $\frac{A'B'}{C'D'} = \frac{AB}{CD}$.

Proof (for readers interested in advanced topics and for graduate students)
The proof is left to an exercise.

The following corollary is the key fact about parallel projection that we will use when we study similarity. In a homework exercise, you will be asked to make a drawing to illustrate the proof.

Theorem 121 (corollary) about lines that are parallel to the base of a triangle in Euclidean Geometry.
In Euclidean Geometry, if line T is parallel to side $\overline{BC}$ of triangle ΔABC and intersects rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points D and E, respectively, then $\frac{AD}{AB} = \frac{AE}{AC}$.

Proof

(1) Suppose that in Euclidean Geometry, line T is parallel to side $\overline{BC}$ of triangle ΔABC and intersects rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points D and E. **(Make a drawing.)**

(2) Let L be line $\overleftrightarrow{AB}$ and let M be line $\overleftrightarrow{AC}$. **(Update your drawing.)**

(3) Let N be the line that passes through point A and is parallel to side $\overline{BC}$. **(Make a new drawing.)**

(4) Consider the parallel projection $Proj_{L,M,T}$ from line L to line M in the direction of line T. By Theorem 120, we know that $\frac{A'D'}{A'B'} = \frac{AD}{AB}$.

(5) But $A' = Proj_{L,M,T}(A) = A$ and $B' = Proj_{L,M,T}(B) = C$ and $D' = Proj_{L,M,T}(D) = E$.

(6) Substituting letters from (5) into the equation from (4), we obtain $\frac{AD}{AB} = \frac{AE}{AC}$.

End of Proof

Our last theorem of the section does not seem to be about parallel projection or parallel lines at all. In fact, it is not. But the proof makes a nice use of the corollary just presented, and also includes some nice review of facts from earlier in the book. You will justify the proof steps in a homework exercise.

Theorem 122 The Angle Bisector Theorem.
In Euclidean Geometry, the bisector of an angle in a triangle splits the opposite side into two segments whose lengths have the same ratio as the two other sides. That is, in ΔABC, if D is the point on side $\overline{AC}$ such that ray $\overrightarrow{BD}$ bisects angle $\angle ABC$, then $\frac{DA}{DC} = \frac{BA}{BC}$.

Proof

(1) Suppose that ΔABC is given in Euclidean Geometry, and that D is the point on side $\overline{AC}$ such that ray $\overrightarrow{BD}$ bisects angle $\angle ABC$. **(Make a drawing.)**

(2) There exists a line L that passes through point C and is parallel to line $\overleftrightarrow{BD}$. **(Justify.) (Make a new drawing.)**

(3) Line $\overleftrightarrow{AB}$ intersects line $\overleftrightarrow{BD}$, and $\overleftrightarrow{BD}$ is parallel to L, so therefore line $\overleftrightarrow{AB}$ must also intersect line L at a point that we can call E. **(Justify) (Make a new drawing.)**

Identify congruent angles and use them to identify congruent segments

(4) $\angle ABD \cong \angle BEC$. **(Justify.) (Make a new drawing.)**

(5) $\angle CBD \cong \angle BCE$. **(Justify.) (Make a new drawing.)**

(6) But $\angle ABD \cong \angle CBD$. **(Make a new drawing.)**

(7) So, $\angle BCE \cong \angle BEC$. **(Make a new drawing.)**

(8) Therefore, $\overline{BE} \cong \overline{BC}$. **(Justify.) (Make a new drawing.)**

Use Parallel Projection.

(9) $\frac{DA}{DC} = \frac{BA}{BE}$ **(Justify.)**

(10) Therefore, $\frac{DA}{DC} = \frac{BA}{BC}$ (by (8) and (9)).

End of proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 10.4 on page 239.

10.2. Similarity

We now turn to the main topic for this chapter: similarity. Before proceeding, it would be very useful for you to read the beginning paragraph of Chapter 7 and then all of Section 7.1.1 and Section 7.1.2, in which the definition of congruence for triangles is developed. The definition of similarity for triangles has the same style.

Definition 80 triangle similarity
To say that two triangles are *similar* means that there exists a correspondence between the vertices of the two triangles and the correspondence has these two properties:

- Each pair of corresponding angles is congruent.
- The ratios of the lengths of each pair of corresponding sides is the same.

If a correspondence between vertices of two triangles has the two properties, then the correspondence is called *a similarity.* That is, the expression *a similarity* refers to a particular correspondence of vertices that has the two properties.

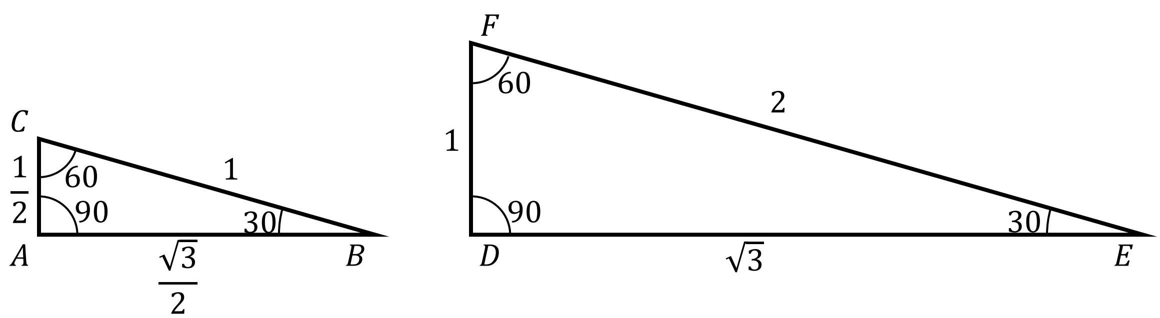
The following statement has a straightforward proof. You will be asked to supply the proof in a homework exercise.

Theorem 123 triangle similarity is an equivalence relation

It is important to discuss notation at this point. It is no accident that Definition 80 above does not include a symbol. There is no commonly-used symbol whose meaning matches the definition of triangle similarity. This may surprise you, because you have all seen the symbol ~ put between triangles. But that symbol means something different, and the difference is subtle. Here is the definition.

Definition 81 symbol for *a similarity* of two triangles
Symbol: $\Delta ABC \sim \Delta DEF$.
Meaning: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ of vertices is a similarity.

There may be more subtlety in the notation than you realize. It is worthwhile to consider a few examples. Refer to the drawing below.



Easy examples involving ΔABC and ΔDEF.

- The statement "ΔABC is similar to ΔDEF" is true. Proof: Consider the correspondence $(A, B, C) \leftrightarrow (D, E, F)$. Observe that $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and $\frac{length(\overline{AB})}{length(\overline{DE})} = \frac{length(\overline{BC})}{length(\overline{EF})} = \frac{length(\overline{CA})}{length(\overline{FD})} = \frac{1}{2}$. Therefore, the correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a similarity. Since there exists a correspondence that is a similarity, we say that the triangles are similar.
- The statement "ΔABC is similar to ΔDFE" is true. Proof: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a similarity. This is the same correspondence from the previous example. Since there exists a correspondence that is a similarity, we say that the triangles are similar.
- The statement "$\Delta ABC \sim \Delta DEF$" is true, because the correspondence $(A, B, C) \leftrightarrow (D, E, F)$ is a similarity.
- The statement "$\Delta ABC \sim \Delta DFE$" is false, because the correspondence $(A, B, C) \leftrightarrow (D, F, E)$ is not a similarity. Observe that $m(\angle ABC) = 30$ while the corresponding angle has measure $m(\angle DFE) = 60$.

We will study four triangle similarity theorems. Before doing that, it is important to digress and review the concept of a triangle congruence theorem.

A digression about triangle congruence theorems

In Section 7.1.2, we saw that the definition of triangle congruence (Definition 54) is that each of the three pairs of corresponding angles is a congruent pair and each of the three pairs of sides is a congruent pair. If all we knew about triangle congruence was the definition, then we would have to verify all six congruences in order to be able to say that a given pair of triangles is congruent. In other words, the concept of triangle congruence would just be a fancy name for the situation where we know that all six pairs of corresponding parts are congruent pairs. That would not be terribly useful. But the Side-Angle-Side (*SAS*) Congruence Axiom <N10> (found in the Axioms for Neutral Geometry in Definition 17 and in the Axioms for Euclidean Geometry in Definition 70) tells us that a certain combination of three congruences is enough to know that two triangles are congruent. More specifically, the axiom states that

> <N10> (*SAS* Axiom) If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

The statement above cannot be proven. It is an *axiom*, a statement that we *assume* is true. It allows us to parlay some known information about two triangles (the fact that two sides and an included angle of one triangle are congruent to the corresponding parts of the other) into some other information (the fact that the other three pairs of corresponding parts are congruent pairs, as well, so that the triangles are congruent). With this axiom, the concept of triangle congruence becomes useful.

Later in Chapter 7, we proved four triangle congruence theorems:

Theorem 54: the *ASA* Congruence Theorem for Neutral Geometry

Theorem 58: the *SSS* congruence theorem for Neutral Geometry

Theorem 70: the Angle-Angle-Side (*AAS*) Congruence Theorem for Neutral Geometry

Theorem 71: the Hypotenuse Leg Congruence Theorem for Neutral Geometry

Each triangle congruence theorem allows us to parlay some known information about two triangles (the fact that three parts of one triangle are congruent to the corresponding parts of the other) into some other information (the fact that the other three pairs of corresponding parts are congruent pairs, as well, so that the triangles are congruent).

We will find that there is analogous situation with similarity and similarity theorems.

End of digression about triangle congruence theorems

So far, all have seen of triangle similarity is the definition:

To say that two triangles are *similar* means that there exists a correspondence between the vertices of two triangles and the correspondence has these two properties:

- Each pair of corresponding angles is congruent.
- The ratios of the lengths of each pair of corresponding sides is the same.

If all we knew about triangle similarity was the definition, then we would have to verify all three angle congruences and check all three ratios of lengths in order to be able to say that a given pair of triangles is similar. In other words, the concept of triangle similarity would just be a fancy name for the situation where we know that each pair of corresponding angles is congruent and the ratios of the lengths of each pair of corresponding sides is the same. That would not be terribly useful.

But it turns out that four triangle similarity theorems can be proven. Each similarity theorem allows us to parlay some known information about two triangles (the fact that some angles of one triangle are congruent to the corresponding angles of the other triangle, or that the ratio of the lengths of some pair of corresponding sides is equal to the ratio of the lengths of some other pair of corresponding sides) into some other information (the fact that every pair of corresponding angles is congruent and the ratios of the lengths of every pair of corresponding sides is the same, so that the triangles are similar). With these theorems, the concept of triangle similarity becomes useful.

We will now discuss the four triangle similarity theorems. We start with the Angle-Angle-Angle Similarity Theorem. In a homework exercise, you will be asked to supply a drawing.

Theorem 124 The Angle-Angle-Angle (*AAA*) Similarity Theorem for Euclidean Geometry
If there is a one-to-one correspondence between the vertices of two triangles, and each pair of corresponding angles is a congruent pair, then the ratios of the lengths of each pair of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

Proof

(1) Suppose that in Euclidean Geometry, triangles ΔABC and ΔDEF are given and that the correspondence of vertices $(A, B, C) \leftrightarrow (D, E, F)$ has the property that $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\angle C \cong \angle F$. **(Make a drawing.)**

Part 1: Build a copy of ΔDEF using vertex A.

(2) There exists a point E' on ray $\overrightarrow{AB}$ such that $\overline{AE'} \cong \overline{DE}$ and there exists a point F' on ray $\overrightarrow{AC}$ such that $\overline{AF'} \cong \overline{DF}$. **(Make a new drawing.)**

(3) $\Delta AE'F' \cong \Delta DEF$ (by (1), (2), and the *SAS* Congruence Axiom, <N10>) (We have built a copy of ΔDEF using vertex A.)

(4) $\angle AE'F' \cong \angle DEF$ (by (3))

(5) $\angle AE'F' \cong \angle ABC$ (by (4) and (1))

(6) Line $\overleftrightarrow{E'F'}$ is parallel to line $\overleftrightarrow{BC}$ (by (5) and Theorem 74)

(7) $\frac{AE'}{AB} = \frac{AF'}{AC}$ (by (6) and Theorem 121)

(8) $\overline{AE'} \cong \overline{DE}$ and $\overline{AF'} \cong \overline{DF}$ (by (2))

(9) $\frac{DE}{AB} = \frac{DF}{AC}$ (by (7) and (8))

Part 2: Build a copy of ΔEFD using vertex B.

(10)-(15) Make the following substitutions in statements (2) through (9):

$$A \to B$$
$$B \to C$$
$$C \to A$$
$$D \to E$$
$$E \to F$$
$$F \to D$$

The result will be the following statement (15) $\frac{EF}{BC} = \frac{ED}{BA}$.

Conclusion

(16) By (9), (15), and transitivity, we have $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$. That is, the ratios of the lengths of each pair of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

End of proof

The following corollary has a very simple proof. You will be asked to supply the proof in the exercises.

Theorem 125 (Corollary) The Angle-Angle (*AA*) Similarity Theorem for Euclidean Geometry
If there is a one-to-one correspondence between the vertices of two triangles, and two pairs of corresponding angles are congruent pairs, then the third pair of corresponding angles is also a congruent pair, and the ratios of the lengths of each pair of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

And the corollary has a simple corollary of its own:

Theorem 126 (corollary) In Euclidean Geometry, the altitude to the hypotenuse of a right triangle creates two smaller triangles that are each similar to the larger triangle.

You will be asked to supply the proof in the exercises.

Our next theorem, the Side-Side-Side Similarity Theorem, is a bit harder to prove than the Angle-Angle-Angle Similarity Theorem. But the general approach is still the same in that a triangle $\Delta AE'F'$ is constructed and then is related to triangles ΔABC and ΔDEF. The proof will use the Angle-Angle Similarity Theorem. In a homework exercise, you will be asked to justify the steps of the proof.

Theorem 127 The Side-Side-Side (*SSS*) Similarity Theorem for Euclidean Geometry
If there is a one-to-one correspondence between the vertices of two triangles, and the ratios of lengths of all three pairs of corresponding sides is the same, then all three pairs of corresponding angles are congruent pairs, so the correspondence is a similarity and the triangles are similar.

Proof

(1) Suppose that in Euclidean Geometry, triangles ΔABC and ΔDEF are given and that the correspondence of vertices $(A, B, C) \leftrightarrow (D, E, F)$ has the property that $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$.
(Make a drawing.)

Part 1: Consider ratios of lengths of sides of $\boldsymbol{\Delta ABC}$ and $\boldsymbol{\Delta DEF}$.

(2) The first part of the string of equalities says that $\frac{DE}{AB} = \frac{EF}{BC}$.

(3) Therefore, $EF = \left(\frac{DE}{AB}\right) BC$.

(4) The second part of the string of equalities says that $\frac{EF}{BC} = \frac{FD}{CA}$.

(5) Therefore, $FD = \left(\frac{EF}{BC}\right) CA$.

Part 2: Build a triangle that is similar to $\boldsymbol{\Delta ABC}$.

(6) There exists a point E' on ray $\overrightarrow{AB}$ such that $\overline{AE'} \cong \overline{DE}$. **(Justify.) (Make a new drawing.)**

(7) There exists a line L that passes through point E' and is parallel to line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(8) Line $\overleftrightarrow{AC}$ intersects line $\overleftrightarrow{BC}$, so line $\overleftrightarrow{AC}$ must also intersect line L at a point that we can call F'. **(Justify.) (Make a new drawing.)**

(9) $\angle AE'F' \cong \angle ABC$. **(Justify.) (Make a new drawing.)**

(10) $\Delta AE'F' \sim \Delta ABC$ **(Justify.) (Make a new drawing.)**

Part 3: Consider ratios of lengths of sides of $\boldsymbol{\Delta ABC}$ and $\boldsymbol{\Delta A'E'F'}$.

(11) We know that $\frac{AE'}{AB} = \frac{E'F'}{BC} = \frac{F'A}{CA}$. **(Justify.)**

(12) The first equality in this string of two equalities says $\frac{AE'}{AB} = \frac{E'F'}{BC}$.

(13) Therefore, $E'F' = \left(\frac{AE'}{AB}\right) BC = \left(\frac{DE}{AB}\right) BC$. (Cross-multiplied then used statement (6))

(14) Conclude that $\overline{E'F'} \cong \overline{EF}$ (by (3) and (13)).

(15) The second equality in the string of two equalities in step (11) says $\frac{E'F'}{BC} = \frac{F'A}{CA}$.

(16) Therefore, $F'A = \left(\frac{E'F'}{BC}\right) CA = \left(\frac{EF}{BC}\right) CA$. (Cross-multiplied then used statement (14))

(17) Conclude that $\overline{F'A} \cong \overline{FD}$ (by (5) and (16)).

Conclusion.

(18) Therefore, $\Delta AE'F' \cong \Delta DEF$ **(Justify.)**

(19) So $\angle A \cong \angle D$ and $\angle F'E'A \cong \angle FED$ **(Justify.)**

(20) But $\angle AE'F' \cong \angle ABC$. **(Justify.)** So $\angle ABC \cong \angle DEF$.

(21) Conclude that $\Delta ABC \sim \Delta DEF$. **(Justify.)**

End of proof

Our final similarity theorem is the Side-Angle-Side (*SAS*) Similarity Theorem. The general approach of the proof again involves the construction of a triangle $\Delta AE'F'$ and then the relationship of this triangle to triangles ΔABC and ΔDEF. Interestingly, the proof will use the *SAS Congruence* Axiom and the *AA* Similarity Theorem. The justification of the steps in the proof is left as an advanced exercises.

Theorem 128 The Side-Angle-Side (*SAS*) Similarity Theorem for Euclidean Geometry
If there is a one-to-one correspondence between the vertices of two triangles, and the ratios of lengths of two pairs of corresponding sides is the same and the corresponding included angles are congruent, then the other two pairs of corresponding angles are also congruent pairs and the ratios of the lengths of all three pairs of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

Proof (for readers interested in advanced topics and for graduate students)

(1) Suppose that in Euclidean Geometry, triangles ΔABC and ΔDEF are given and that the correspondence of vertices $(A,B,C) \leftrightarrow (D,E,F)$ has the property that $\angle A \cong \angle D$ and $\frac{DE}{AB} = \frac{DF}{AC}$. **(Make a drawing.)**

Introduce line L and point F'.

(2) There exists a point E' on ray $\overrightarrow{AB}$ such that $\overline{AE'} \cong \overline{DE}$. **(Justify.) (Make a new drawing.)**

(3) There exists a line L that passes through point E' and is parallel to line $\overleftrightarrow{BC}$. **(Justify.) (Make a new drawing.)**

(4) Line $\overleftrightarrow{AC}$ intersects line $\overleftrightarrow{BC}$, so line $\overleftrightarrow{AC}$ must also intersect line L at a point that we can call F'. **(Justify.) (Make a new drawing.)**

Use AA Similarity to show that $\Delta AE'F' \sim \Delta ABC$.

(5) $\angle AE'F' \cong \angle ABC$. **(Justify.) (Make a new drawing.)**

(6) $\Delta AE'F' \sim \Delta ABC$. **(Justify.) (Make a new drawing.)**

Use Parallel Projection and the *SAS* Congruence Axiom to show that $\Delta AE'F' \cong \Delta DEF$.

(7) $\frac{AF'}{AC} = \frac{AE'}{AB}$. **(Justify.)**

(8) $AF' = \frac{AE' \cdot AC}{AB}$. **(Justify.)**

(9) $AF' = \left(\frac{AC}{AB}\right) DE$. **(Justify.)**

(10) But $DF = \left(\frac{AC}{AB}\right) DE$. **(Justify.)**

(11) Therfore, $\overline{AF'} \cong \overline{DF}$. **(Justify.)**

(12) So $\Delta AE'F' \cong \Delta DEF$. **(Justify.)**

Use AA Similarity to show that $\Delta ABC \sim \Delta DEF$.

(13) $\angle AE'F' \cong \angle DEF$. **(Justify.)**

(14) But also $\angle AE'F' \cong \angle ABC$. **(Justify.)**

(15) Therefore, $\angle ABC \cong \angle DEF$. **(Justify.)**

(16) So $\Delta ABC \sim \Delta DEF$. **(Justify.)**

End of Proof

The four so-called *similarity theorems* that we studied above all have the same sort of statement. Each has as its hypothesis some known information about two triangles (the fact that some angles of one triangle are congruent to the corresponding angles of the other triangle, or that the ratio of the lengths of some pair of corresponding sides is equal to the ratio of the lengths of some other pair of corresponding sides) and has as its conclusion the statement that the two triangles are similar.

Our final theorem of the section is about similarity but is of a very different style. The first theorem shows that in similar triangles, it is not just the pairs of corresponding sides whose ratios are the same. There are many line segments associated with triangles, and most of these will have the same ratio behavior as the sides in similar triangles. The following theorem mentions three kinds of line segments: altitudes, angle bisectors, and medians.

Theorem 129 About the ratios of lengths of certain line segments associated to similar triangles in Euclidean Geometry.

In Euclidean Geometry, if $\Delta \sim \Delta'$, then

$$\frac{length\ of\ side}{length\ of\ side'} = \frac{length\ of\ altitude}{length\ of\ altitude'} = \frac{length\ of\ angle\ bisector}{length\ of\ angle\ bisector'} = \frac{length\ of\ median}{length\ of\ median'}$$

You will be asked to supply the proof in a homework exercise.

So far, we have seen a bunch of theorems about similarity, but we have not seen any examples of the use of similarity. You will work on couple in the exercises. But our most important application of similarity comes in the coming sections, when we use similarity to prove the Pythagorean Theorem and also prove a theorem about the product of $base \cdot height$ in a triangle.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 10.4 on page 239.

10.3. Applications of Similarity

In this section, we will study two very important applications of similarity. Most people remember both. The first that we will study is The Pythagorean Theorem.

Theorem 130 The Pythagorean Theorem of Euclidean Geometry
In Euclidean Geometry, the sum of the squares of the length of the two sides of any right triangle equals the square of the length of the hypotenuse. That is, in Euclidean Geometry, given triangle ΔABC with $a = BC$ and $b = CA$ and $c = AB$, if angle $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.

There are hundreds of proofs of the Pythagorean Theorem. The most beautiful proofs make use of simple pictures involving triangles and squares, and discuss the sums of various areas. But in this book, we have not yet discussed area. However, we have discussed similarity. Here is a proof that uses the concept of similarity.

Proof

(1) In Euclidean Geometry, suppose that triangle ΔABC has a right angle at C and that $a = BC$ and $b = CA$ and $c = AB$. **(Make a drawing.)**
(2) Let D be the foot of the altitude drawn from vertex C. That is, D is the point on side $\overline{AB}$ such that segment $\overline{CD}$ is perpendicular to side $\overline{AB}$. Let $x = AD$ and $y = BD$. **(Make a new drawing.)**
(3) $\Delta ADC \sim \Delta ACB$. **(Justify) (Make a new drawing.)**
(4) $\frac{x}{b} = \frac{b}{c}$. **(Justify)**
(5) $cx = b^2$. **(Justify)**
(6) $\Delta BDC \sim \Delta BCA$. **(Justify) (Make a new drawing.)**
(7) $\frac{y}{a} = \frac{a}{c}$. **(Justify)**
(8) $cy = a^2$. **(Justify)**
(9) $a^2 + b^2 = cx + cy$. **(Justify)**
(10) $a^2 + b^2 = c(x + y)$. **(arithmetic)**
(11) $a^2 + b^2 = c^2$. **(Justify)**

End

Recall that statement of the form $If\ P\ then\ Q$ is called a *conditional statemen*t. The *converse statement* is $If\ Q\ then\ P$. Remember also that the converse statement does not mean the same thing as the original statement and so in general, the fact that a conditional statement is true does not mean that its converse is true. For example, the conditional statement $If\ x = -3\ then\ x^2 =$

9 is true, but the converse statement $If\ x^2 = 9\ then\ x = -3$ is false. But in the case of the Pythagorean Theorem for Euclidean Geometry, the converse statement is also a theorem. You will justify the proof steps in a homework exercise.

Theorem 131 The Converse of the Pythagorean Theorem of Euclidean Geometry
In Euclidean Geometry, if the sum of the squares of the length of two sides of a triangle equals the square of the length of the third side, then the angle opposite the third side is a right angle. That is, in Euclidean Geometry, given triangle ΔABC with $a = BC$ and $b = CA$ and $c = AB$, if $a^2 + b^2 = c^2$, then angle $\angle C$ is a right angle.

Proof

(1) In Euclidean Geometry, suppose that triangle ΔABC is given and that $a = BC$ and $b = CA$ and $c = AB$ and that $a^2 + b^2 = c^2$. **(Make a drawing.)**

(2) There exist three points D, E, F such that $\angle EFD$ is a right angle and such that $\overline{EF} \cong \overline{BC}$ and $\overline{FD} \cong \overline{CA}$. **(Justify. If you do this properly, citing axioms and theorems, it will take quite a few steps.) (Make a new drawing.)**

(3) Observe that $EF = BC = a$ and $FD = CA = b$. Therefore, $(DE)^2 = a^2 + b^2$. (by the Pythagorean Theorem Theorem 130 applied to triangle ΔDEF.)

(4) Thus $(DE)^2 = c^2$ (by (3), (1), and transitivity), so $\overline{DE} \cong \overline{AB}$..

(5) Therefore, $\Delta DEF \cong \Delta ABC$ **(Justify.)**

(6) Therefore, $\angle EFD \cong \angle BCA$. That is, $\angle BCA$ must be a right angle (by (5) and Definition 54 of triangle congruence).

End

The proof of the Pythagorean Theorem used the AA Similarity Theorem. The second important application the AA Similarity theorem that we will study has to do with the product of $base \cdot height$ in a triangle. We should be clear about the terminology.

Definition 82 base times height
For each side of a triangle, there is an opposite vertex, and there is an altitude segment drawn from that opposite vertex. The expression "$base\ times\ height$" or "$base \cdot height$" refers to the product of the length of a side of a triangle and the length of the corresponding altitude segment drawn to that side. The expression can be abbreviated $b \cdot h$.

An obvious question is, does it matter which side of the triangle is chosen to be the base? The answer is no. That is, the value of the product $b \cdot h$ does not depend on which side of the triangle is chosen to be the base.

Theorem 132 In Euclidean Geometry, the product of $base \cdot height$ in a triangle does not depend on which side of the triangle is chosen as the base.

The proof is a straightforward application of the AA Similarity Theorem (Theorem 125), but it is rather tedious to follow. You will be asked to justify the steps in a homework exercise.

Proof

(1) Suppose that a triangle is given in Euclidean Geometry. There are two possibilities: either the triangle is a right triangle, or it is not.

Case 1: Right triangle

(2) If the triangle is a right triangle, label the vertices A, B, C so that the right-angle is at C. Let $a = BC$ and $b = CA$ and $c = AB$. **(Make a drawing.)**

Observe that when side $\overline{BC}$ is chosen as the base, then the product of $base \cdot height$ is $a \cdot b$, and when side $\overline{CA}$ is chosen as the base, then the product of $base \cdot height$ is $b \cdot a$. These two products are equal. But we need to see what happens when side $\overline{AB}$ is chosen as the base.

(3) Let F be the point on side $\overline{AB}$ such that segment $\overline{CF}$ is perpendicular to side $\overline{AB}$, and let $z = CF$. **(Make a new drawing.)**

(4) $\Delta ABC \sim \Delta ACF$. **(Justify.) (Make a new drawing.)**

(5) $\frac{a}{z} = \frac{c}{b}$. **(Justify)**

(6) Then $a \cdot b = c \cdot z$. **(Justify.)**

Conclusion of Case 1

(7) We see that the product of $base \cdot height$ does not depend on the choice of base in this case.

Case 2: Not a right triangle

(8) If the triangle is not a right triangle, label the vertices A, B, C. **(Make a new drawing.)**

Let D be the point on side $\overline{BC}$ such that segment $\overline{AD}$ is perpendicular to side $\overline{BC}$, and let $x = AD$. **(Update your drawing.)**

Let E be the point on side $\overline{CA}$ such that segment $\overline{BE}$ is perpendicular to side $\overline{CA}$, and let $y = BE$. **(Update your drawing.)**

Let F be the point on side $\overline{AB}$ such that segment $\overline{CF}$ is perpendicular to side $\overline{AB}$, and let $z = CF$. **(Update your drawing.)**

(9) $\Delta ACD \sim \Delta BCE$. **(Justify.) (Make a new drawing.)**

(10) $\frac{b}{a} = \frac{x}{y}$. **(Justify)**

(11) Then $a \cdot x = b \cdot y$. **(Justify.)**

(12) $\Delta BAE \sim \Delta CAF$. **(Justify.) (Make a new drawing.)**

(13) $\frac{c}{b} = \frac{y}{z}$. **(Justify)**

(14) Then $b \cdot y = c \cdot z$. **(Justify.)**

Conclusion of Case 2

(15) We see that the product of $base \cdot height$ does not depend on the choice of base in this case, either.

Conclusion

(16) The product of $base \cdot height$ does not depend on the choice of base in either case.

End of proof

Theorem 132 will play an extremely important role in the next chapter, when we introduce the concept of Area.

10.4. Exercises for Chapter 10

Exercises for Section 10.1 (Parallel Projections) (Section starts on page 225)

[1] (Advanced) Prove Theorem 118 (Parallel Projection in Euclidean Geometry preserves betweenness.) (found on page 228).

[2] (Advanced) Prove Theorem 118 (Parallel Projection in Euclidean Geometry preserves betweenness.) (found on page 228).

[3] Justify the steps and make drawings to illustrate the proof of Theorem 119 (Parallel Projection in Euclidean Geometry preserves congruence of segments.) (found on page 228).

[4] (Advanced) Prove Theorem 120 (Parallel Projection in Euclidean Geometry preserves ratios of lengths of segments.) (found on page 229).

[5] Make a drawing to illustrate the proof of Theorem 121 ((corollary) about lines that are parallel to the base of a triangle in Euclidean Geometry.) (found on page 229).

[6] Justify the steps in the proof of Theorem 122 (The Angle Bisector Theorem.) (found on page 230).

Exercises for Section 10.2 (Similarity) (Section starts on page230)

[7] Prove Theorem 123 (triangle similarity is an equivalence relation) (found on page 231).

[8] Make a drawing to illustrate the proof of Theorem 124 (The Angle-Angle-Angle (*AAA*) Similarity Theorem for Euclidean Geometry) (found on page 233).

[9] Prove Theorem 125 ((Corollary) The Angle-Angle (*AA*) Similarity Theorem for Euclidean Geometry) (found on page 234).

[10] Prove Theorem 126 ((corollary) In Euclidean Geometry, the altitude to the hypotenuse of a right triangle creates two smaller triangles that are each similar to the larger triangle.) (found on page 234).

[11] Justify the steps in the proof of Theorem 127 (The Side-Side-Side (*SSS*) Similarity Theorem for Euclidean Geometry) (found on page 234).

[12] (Advanced.) Justify the steps in the proof of Theorem 128 (The Side-Angle-Side (*SAS*) Similarity Theorem for Euclidean Geometry) (found on page 235).

[13] There is an *ASA* Congruence Theorem (Theorem 54). Why isn't there an *ASA* Similarity Theorem? Explain.

[14] Prove Theorem 129 (About the ratios of lengths of certain line segments associated to similar triangles in Euclidean Geometry.) (found on page 236).
Hint: Do the proof in three parts, as follows.
Part I: Suppose that a pair of similar triangles is given in Euclidean Geometry and that a pair of corresponding altitudes is chosen. Label the vertices of the triangles A, B, C and E, F, G so that the chosen altitudes are segments $\overline{AD}$ and $\overline{EH}$. Consider triangles ΔABD and ΔEFH. Show that

$$\frac{AB}{EF} = \frac{AD}{EH}$$

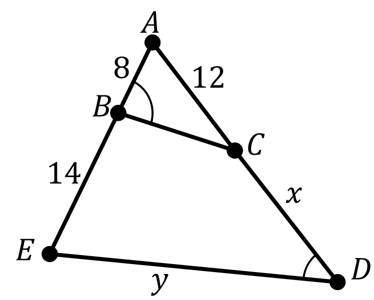
Part II: Suppose that a pair of similar triangles is given in Euclidean Geometry and that a pair of corresponding angle bisectors is chosen. Label the vertices of the triangles A, B, C and E, F, G so that the chosen angle bisectors are segments $\overline{AD}$ and $\overline{EH}$. Consider triangles ΔABD and ΔEFH. Show that

$$\frac{AB}{EF} = \frac{AD}{EH}$$

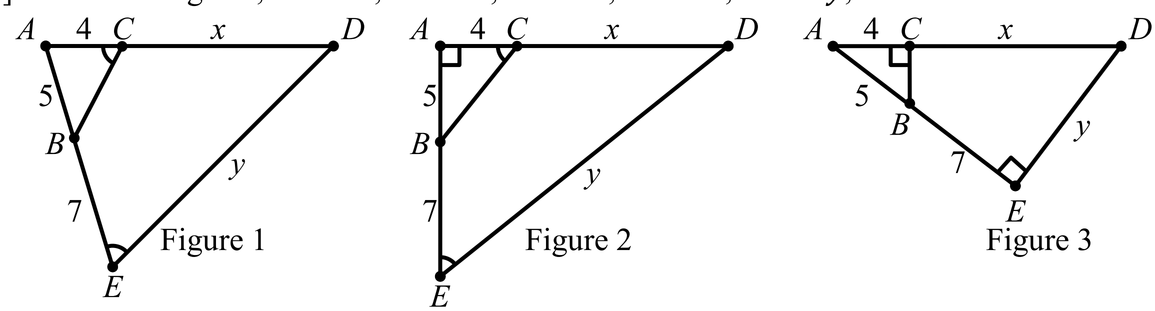
Part III: Suppose that a pair of similar triangles is given in Euclidean Geometry and that a pair of corresponding medians is chosen. Label the vertices of the triangles A, B, C and E, F, G so that the chosen medians are segments $\overline{AD}$ and $\overline{EH}$. Consider triangles ΔABD and ΔEFH. Show that

$$\frac{AB}{EF} = \frac{AD}{EH}$$

[15] In the figure at right, is it possible to determine x? Is it possible to determine y? Explain.



[16] In all three figures, $AB = 5$, $AC = 4$, $BE = 7$, $CD = x$, $DE = y$, and $\angle ACB \cong \angle AED$



(A) In Figure 1, identify two similar triangles and explain how you know that they are similar. Draw them with matching orientations.

(B) Find the value of x for Figure 1. Observe that this will be the value of x for all three figures.

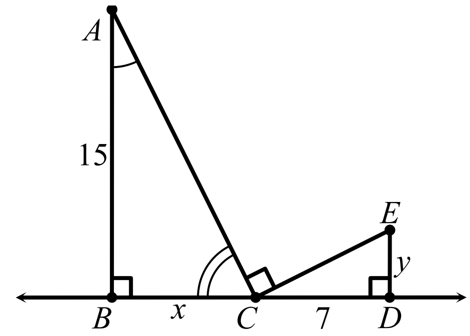
(C) Figure 2 is a special case of Figure 1, the special case in which $\angle BAC$ is a right angle. Find y by using the Pythagorean Theorem and the known value of x from part (B).

(D) Figure 3 is a different special case of Figure 1, the special case in which $\angle ACB$ is a right angle. Find y by using the Pythagorean Theorem and the known value of x from part (B).

Your answers to [16](C) and (D) should differ. This proves that it is not possible to find the value of y for Figure 1 using only the information shown in Figure 1. More information is needed, such as the additional information given in Figure 2 or Figure 3.

[17] Refer to the drawing at right. Find y in terms of x.

Hint: Start by identifying two similar triangles. Be sure to explain how you know that they are similar, and be sure to draw the triangles side-by-side with the same orientation and with all known parts labeled.



Exercises for Section 10.3 (Applications of Similarity) (Section starts on page 237)

[18] Justify the steps in the proof of Theorem 130 (The Pythagorean Theorem of Euclidean Geometry) (found on page 237).

[19] The Hypotenuse Leg Theorem (Theorem 71) is a theorem of Neutral Geometry. That means that it can be proven using only the Neutral Geometry Axioms (Definition 17) and is therefore true in both Neutral Geometry and Euclidean Geometry. The proof of the theorem using only the Neutral Axioms is fairly difficult. (See the proof of Theorem 71, found on page 183.) The theorem can be proved much more easily using the Pythagorean Theorem. Such a proof would prove that the theorem is true in Euclidean Geometry, but it would not prove that the theorem is true in Neutral Geometry. Prove the Hypotenuse-Leg Theorem using the Pythagorean Theorem.

[20] Justify the steps in the proof of Theorem 131 (The Converse of the Pythagorean Theorem of Euclidean Geometry) (found on page 238).

[21] Justify the steps in the proof of Theorem 132 (In Euclidean Geometry, the product of $base \cdot height$ in a triangle does not depend on which side of the triangle is chosen as the base.)

[22] You studied a proof of Theorem 132 in the previous exercise. Here is an invalid proof of that theorem. What is wrong with it? Explain.

Proof

(1) Suppose that ΔABC is given in Euclidean Geometry, and that segments $\overline{AD}$ and $\overline{BE}$ and $\overline{CF}$ are altitudes.

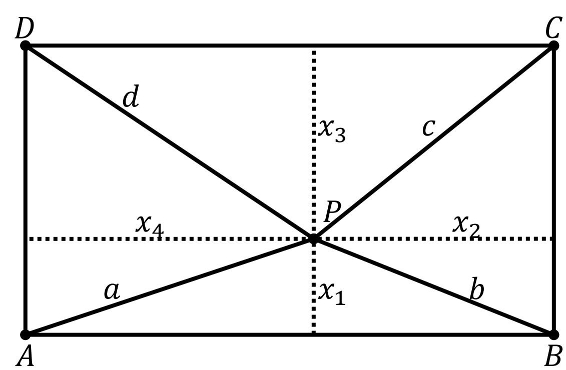
(2) Let $a = BC$ and $b = CA$ and $c = AB$ and $x = CF$ and $y = AD$ and $z = BE$.

(3) Then $area(\Delta ABC) = \frac{a \cdot x}{2}$ and $area(\Delta ABC) = \frac{b \cdot y}{2}$ and $area(\Delta ABC) = \frac{c \cdot z}{2}$.

(4) Therefore, $\frac{a \cdot x}{2} = \frac{b \cdot y}{2} = \frac{c \cdot z}{2}$, so $a \cdot x = b \cdot y = c \cdot z$.

End of Proof

[23] Prove that in Euclidean Geometry, if P is a point in the interior of $Rectangle(ABCD)$ and a, b, c, d are the lengths of segments $\overline{AP}, \overline{BP}, \overline{CP}, \overline{DP}$, then $a^2 + c^2 = b^2 + d^2$. Hint: Let x_1, x_2, x_3, x_4 be the distances from P to lines $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CD}, \overleftrightarrow{DA}$. Express a^2, b^2, c^2, d^2 in terms of x_1, x_2, x_3, x_4. Then build the expressions $a^2 + c^2$ and $b^2 + d^2$ and compare them.



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11. Euclidean Geometry III: Area

So far in this book, there has been no development of a concept of area. Our eleven axioms of Euclidean Geometry (Definition 70, found on page 209) do not mention area, and it has not been the subject of any theorem or definition. We would like to have a notion of area for our abstract geometry that mimics our notion of area for drawings. But before we can do that, we need to more precisely articulate what we are trying to mimic.

What do we know about computing area in drawings? To compute the area of certain simple shapes, we measure certain lengths and substitute those numbers into formulas, depending on the shapes. The list of shapes for which we have area formulas is a very short list. To compute the area of a more complicated shape, we subdivide the shape into simple shapes and then add up the area of the simple shapes. We assume that different subdivisions of a shape will give the same area.

To mimic this in our abstract geometry, we will need the following

(1) a short list of simple regions whose areas we know how to compute by formulas
(2) a description of more general regions and the procedure for subdividing them
(3) a definition of the area of a region as the sum of the areas of simple regions, and a verification that different subdivisions of a region will give the same area

Previously in this book, we have studied the concepts of measuring distance and measuring angles. In both cases, we were trying to mimic the use of certain tools in drawings—rulers and protractors—and we found that the terminology of functions could make our writing precise. We will use the terminology of functions in our formulation of the concept of area.

11.1. Triangular Regions, Polygons, and Polygonal Regions

We start with item (1), the short list of simple regions whose areas we know how to compute by formulas. It is indeed a very short list: triangular regions.

Recall that a triangle is defined to be the union of three line segments determined by three non-collinear points (Definition 28, found on page 104). The interior of a triangle is defined in terms of the intersection of three half-planes (Definition 38 found on page 123). We will use these in our definition of a triangular region.

Definition 83 triangular region, interior of a triangular region, boundary of a triangular region

Symbol: $\blacktriangle ABC$
Spoken: triangular region A, B, C
Usage: A, B, C are non-collinear points
Meaning: the union of triangle ΔABC and the interior of triangle ΔABC. In symbols, we would write $\blacktriangle ABC = \Delta ABC \cup Interior(\Delta ABC)$.
Additional Terminology: the interior of a triangular region is defined to be the interior of the associated triangle. That is, $Interior(\blacktriangle ABC) = Interior(\Delta ABC)$. The boundary of

a triangular region is defined to be the associated triangle, itself. That is, $Boundary(\blacktriangle ABC) = \Delta ABC$.

We are going to define the area of a triangular region to be $\frac{1}{2} base \cdot height$, or $\frac{bh}{2}$ for short. But we should be more thorough. First, we should confirm that the value that we get for the area will not depend on the choice of base. Recall that Theorem 132 (found on page 238)states that the product of $base \cdot height$ in a triangle does not depend on choice of base. Secondly, we should use the terminology of functions to make our definition of area more precise. We apply the formula $\frac{bh}{2}$ to a triangular region and we get a positive real number as a result. So the process of applying the formula can be described as a function whose domain is the set of all triangular regions and whose codomain is the set of non-negative real numbers. Here is a symbol that we can use for the set of all triangular regions.

Definition 84 the set of all triangular regions is denoted by $\mathcal{R}_{\blacktriangle}$.

We will use that symbol in the definition of area for triangular regions.

Definition 85 the area function for triangular regions

symbol: $Area_{\blacktriangle}$

spoken: the area function for triangular regions

meaning: the function $Area_{\blacktriangle}: \mathcal{R}_{\blacktriangle} \to \mathbb{R}^+$ defined by $Area_{\blacktriangle}(\blacktriangle ABC) = \frac{bh}{2}$, where b is the length of any side of ΔABC and h is the length of the corresponding altitude segment. (Theorem 132 guarantees that the resulting value does not depend on the choice of base.)

We now move on to item (2) on the list at the start of this chapter: a description of more general regions and the procedure for subdividing them. We will see how triangular regions can be assembled to create what will be called *polygonal regions*. The areas of the triangular regions will be used to find the areas of the polygonal regions.

Quadrilaterals were introduced in Definition 39 (found on page 129). The definition of Polygons will be analogous.

Definition 86 polygon

words: polygon $P_1, P_2, \dots, P_n$

symbol: $Polygon(P_1P_2 \dots P_n)$

usage: $P_1, P_2, \dots, P_n$ are distinct points, with no three in a row being collinear, and such that the segments $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_nP_1}$ intersect only at their endpoints.

meaning: $Polygon(P_1P_2 \dots P_n)$ is defined to be the following set:

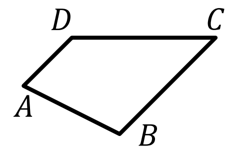
$$Polygon(P_1P_2 \dots P_n) = \overline{P_1P_2} \cup \overline{P_2P_3} \cup \dots \cup \overline{P_nP_1}$$

additional terminology: Points $P_1, P_2, \dots, P_n$ are each called a *vertex* of the polygon. Pairs of vertices of the form $\{P_k, P_{k+1}\}$ and the pair $\{P_n, P_1\}$ are called adjacent vertices. The n segments $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_nP_1}$ whose endpoints are adjacent vertices are each called a *side* of the polygon. Segments whose endpoints are non-adjacent vertices are each called a *diagonal* of the polygon.

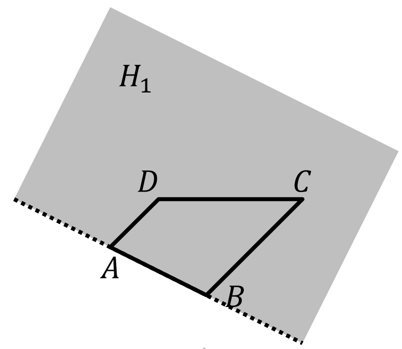
We are interested in defining something called a *polygonal region*. We would hope that it could be done the same way that we defined a triangular region. That is, define the interior of a polygon, and then define a polygonal region to be the union of a polygon and its interior.

But defining the interior of a polygon is tricky. In some cases, the interior could be defined in the same way that we defined the interior of a triangle.

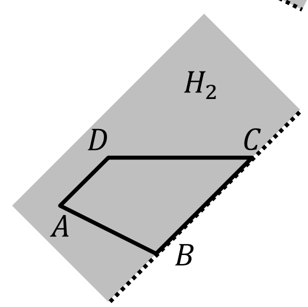
For example, consider the quadrilateral $Polygon[A, B, C, D]$ shown at right.



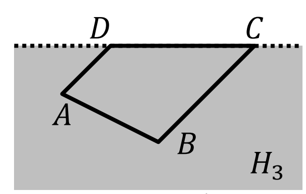
Let H_1 be the half-plane bordered by line $\overleftrightarrow{AB}$ and containing point C. (The half-plane stops at the dotted line but extends forever in the other directions.)



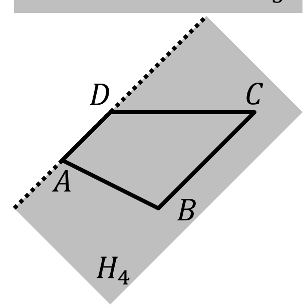
Let H_2 be the half-plane bordered by line $\overleftrightarrow{BC}$ and containing point D.



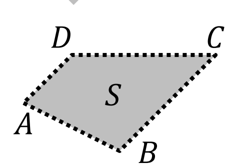
Let H_3 be the half-plane bordered by line $\overleftrightarrow{CD}$ and containing point A.



Let H_4 be the half-plane bordered by line $\overleftrightarrow{DA}$ and containing point B.

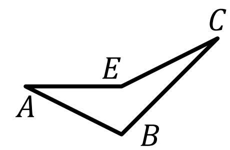


Let set S be the intersection $S = H_1 \cap H_2 \cap H_3 \cap H_4$. Then S is the shaded region shown at right.

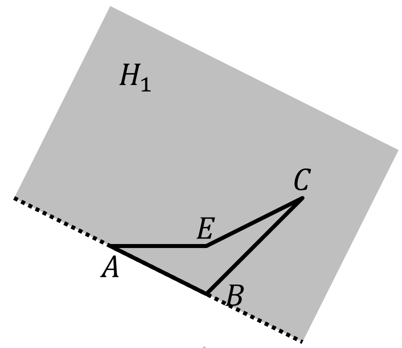


For the quadrilateral $Polygon[A, B, C, D]$, we could define the interior to be the intersection of the four half-planes, as shown.
But sometimes the intersection of half-planes does not turn out to be the set that we would think of as the "inside" of a polygon.

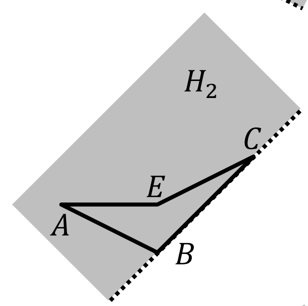
For example, consider the quadrilateral $Polygon[A, B, C, E]$ shown at right.



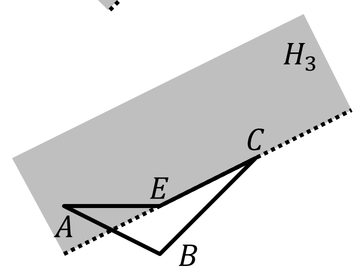
Let H_1 be the half-plane bordered by line $\overleftrightarrow{AB}$ and containing point C. (The half-plane stops at the dotted line but extends forever in the other directions.)



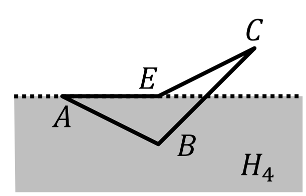
Let H_2 be the half-plane bordered by line $\overleftrightarrow{BC}$ and containing point E.



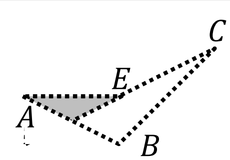
Let H_3 be the half-plane bordered by line $\overleftrightarrow{CE}$ and containing point A.



Let H_4 be the half-plane bordered by line $\overleftrightarrow{EA}$ and containing point B.



Let set S be the intersection $S = H_1 \cap H_2 \cap H_3 \cap H_4$. Then S is the shaded region shown at right.



We see that the set S is not the whole region "inside" $Polygon[A, B, C, E]$.

The problem is that $Polygon[A, B, C, E]$ is not convex. We have seen a definition of convex quadrilateral (Definition 40, found on page 132), but we have not seen a definition of convex polygon. Here is a definition.

Definition 87 convex polygon
A *convex* polygon is one in which all the vertices that are not the endpoints of a given side lie in the same half-plane determined by that side. A polygon that does not have this property is called *non-convex*.

In the two examples above, we see that

$Polygon[A, B, C, D]$ is convex because the following four statements are all true:

Vertices C, D lie in the same half-plane H_1 determined by line $\overleftrightarrow{AB}$.
Vertices D, A lie in the same half-plane H_2 determined by line $\overleftrightarrow{BC}$.
Vertices A, B lie in the same half-plane H_3 determined by line $\overleftrightarrow{CD}$.
Vertices B, C lie in the same half-plane H_4 determined by line $\overleftrightarrow{DA}$.

$Polygon[A, B, C, E]$ is non-convex because:

Vertices A, B do not both lie in the same half-plane H_3 determined by line $\overleftrightarrow{CE}$.
Vertices B, C do not both lie in the same half-plane H_4 determined by line $\overleftrightarrow{EA}$.

It is the existence of non-convex polygons that makes it difficult to state a simple definition of the interior of a polygon. So we will not try to state a simple defintion of the interior of a polygon. Instead, we will skip to the concept of a *polygonal region*.

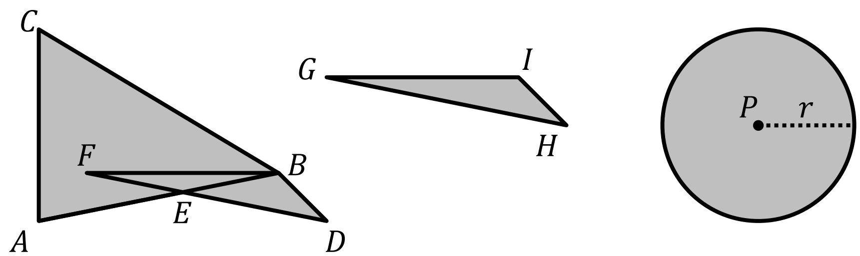
Definition 88 complex, polygonal region, separated, connected polygonal regions
A *complex* is a finite set of triangular regions whose interiors do not intersect. That is, a set of the form $C = \{\blacktriangle_1, \blacktriangle_2, \dots, \blacktriangle_k\}$ where each $\blacktriangle_i$ is a triangular region and such that if $i \neq j$, then the intersection $Interior(\blacktriangle_i) \cap Interior\left(\blacktriangle_j\right)$ is the empty set.

A *polygonal region* is a set of points that can be described as the union of the triangular regions in a complex. That is a set of the form

$$R = \blacktriangle_1 \cup \blacktriangle_2 \cup \dots \cup \blacktriangle_k = \bigcup_{i=1}^{k} \blacktriangle_i$$

We say that a polygonal region can be *separated* if it can be written as the union of two disjoint polygonal regions. A *connected polygonal region* is one that cannot be separated into two disjoint polygonal regions. We will often use notation like $Region(P_1P_2 \dots P_n)$ to denote a connected polygonal region. In that symbol, the letters $P_1, P_2, \dots, P_n$ are vertices of the region (I won't give a precise definition of vertex. You get the idea.)

For example, in the figure shown below, the set $\{\blacktriangle ABC, \blacktriangle BDE\}$ is a complex, but the set $\{\blacktriangle ABC, \blacktriangle BDF\}$ is not a complex, because the interiors of $\blacktriangle ABC$ and $\blacktriangle BDF$ intersect.



The set $S = \blacktriangle ABC \cup \blacktriangle BDE$ is a polygonal region because it is possible to write S as the union of the triangular regions in a complex. The symbol $Region(AEDBC)$ could be used to denote polygonal region S.

Of course it is also possible to write S as the union of triangular regions that overlap. For example, we can write $S = \blacktriangle ABC \cup \blacktriangle BDF$. This does not disqualify S from being called a polygonal region. The fact that a complex exists for set S qualifies S to be called a polygonal region.

Also note that the set the set

$$\{\blacktriangle ABC, \blacktriangle BDE, \blacktriangle GHI\}$$

is a complex. Therefore, the set

$$T = \blacktriangle ABC \cup \blacktriangle BDE \cup \blacktriangle GHI$$

is a polygonal region. It is a polygonal region that can be separated. Here is a separation:

$$T = S \cup \blacktriangle GHI$$

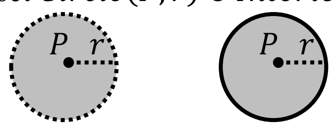
So T is a polygonal region but it is not a connected polygonal region. Clearly, a symbol like $Region(AEDBCGHI)$ would be a terrible choice to describe polygonal region T. That's why the definition above states that we will only use that sort of notation only for connected polygonal regions. For example, we could write $T = Region(AEDBC) \cup \blacktriangle GHI$.

So far, it seems like every "filled-in" shape is a polygonal region. But this is not true. In the picture above, the set consisting of the union of $Circle(P, r)$ and its interior is not a polygonal region.

It is not very hard to come up with a definition for the interior of a polygonal region. But it helps if we first define open disks.

Definition 89 open disk, closed disk
symbol: $disk(P, r)$
spoken: the open disk centered at point P with radius r.
meaning: the set $Interior\big(Circle(P, r)\big)$. That is, the set $\{Q: distance(P, Q) < r\}$.
another symbol: $\overline{disk(P, r)}$
spoken: the closed disk centered at point P with radius r.

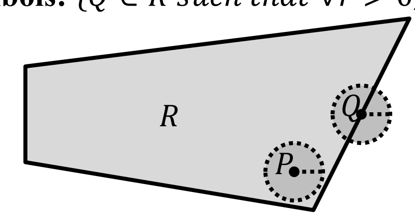
meaning: the set $Circle(P,r) \cup Interior\big(Circle(P,r)\big)$. That is, $\{Q: distance(P,Q) \leq r\}$.
pictures:



the open disk $disk(P,r)$ the closed disk $\overline{disk(P,r)}$

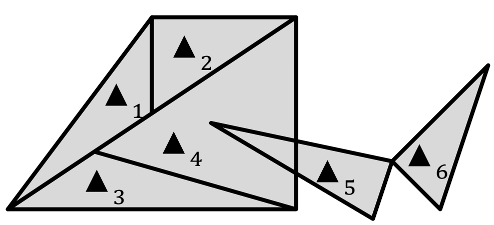
Now we can easily define the interior of a polygonal region.

Definition 90 interior of a polygonal region, boundary of a polygonal region
words: the interior of polygonal region R
meaning: the set of all points P in R with the property that there exists some open disk centered at point P that is entirely contained in R
meaning in symbols: $\{P \in R \; such \; that \; \exists r > 0 \; such \; that \; disk(P,r) \subset R\}$
additional terminology: the boundary of polygonal region R
meaning: the set of all points Q in R with the property that no open disk centered at point Q is entirely contained in R. This implies that every open disk centered at point Q contains some points that are not elements of the region R.
meaning in symbols: $\{Q \in R \; such \; that \; \forall r > 0, disk(P,r) \not\subset R\}$
picture:



P is an interior point; Q is a boundary point

Now that we have a definition of the interior of a polygonal region, we should discuss what happens when we take the union of two polygonal regions, because the interiors play a role in the answer. Consider the drawing below.



Here are some small polygonal regions:

The set $R_1 = \blacktriangle_1 \cup \blacktriangle_2$ is a polygonal region with complex $C_1 = \{\blacktriangle_1, \blacktriangle_2\}$.
The set $R_2 = \blacktriangle_3 \cup \blacktriangle_4$ is a polygonal region with complex $C_2 = \{\blacktriangle_3, \blacktriangle_4\}$.
The set $R_3 = \blacktriangle_5 \cup \blacktriangle_6$ is a polygonal region with complex $C_3 = \{\blacktriangle_5, \blacktriangle_6\}$.

We can combine them into larger regions by forming their set unions:

The set $R_1 \cup R_2$ is a polygonal region with complex $C = C_1 \cup C_2 = \{\blacktriangle_1, \blacktriangle_2, \blacktriangle_3, \blacktriangle_4\}$.
The set $R_2 \cup R_3$ is a polygonal region but the set $\{\blacktriangle_3, \blacktriangle_4, \blacktriangle_5, \blacktriangle_6\}$ is not its complex.

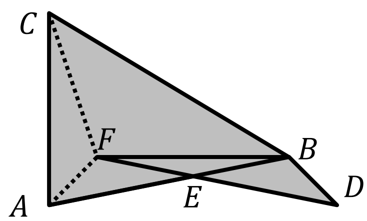
From this example, we would infer that the union of two polygonal regions is a new polygonal region. If the interiors of the two polygonal regions do not intersect, then a complex for the new polygonal region can be obtained by taking the union of complexes for the two regions. But if the interiors of the two polygonal regions do intersect, then the union of their complexes might not be a complex for the new region. This issue will come up when we consider the area of the union of two polynomial regions.

Finally we are ready to move on to item (3) on the list at the start of the chapter: a definition of the area of a region as the sum of the areas of simple regions, and a verification that different subdivisions of a region will give the same area.

11.2. The Area of a Polygonal Region

With the terminology that we have developed, it is fairly easy to state a definition for the area of a polygonal region. We want to say that the area of a polygonal region R is defined to be the sum of the areas of the triangular regions in a complex for R.

But there is a potential problem, because for any polygonal region there are many complexes. Recall $S = Region(AEDBC)$ in our earlier example. With some dotted lines added as shown, we can see a few obvious complexes for S.



$$C_1 = \{\blacktriangle ABC, \blacktriangle BDE\}$$
$$C_2 = \{\blacktriangle ABF, \blacktriangle BCF, \blacktriangle CAF, \blacktriangle BDE\}$$
$$C_3 = \{\blacktriangle AE, \blacktriangle DBF, \blacktriangle BCF, \blacktriangle CAF\}$$

Are we sure that the sum of the areas of the triangular regions in complex C_1 will be the same as the sum of the areas of the triangular regions in complex C_2? It would not be hard to show that in this diagram, the sum of the areas would be the same for complexes C_1, C_2, C_3. But there are lots of other complexes for region S. And there are lots of other poygonal regions. We need a general theorem that will settle the question once and for all. A general theorem *is* possible, and can be proven with a proof at the level of this course. We will accept the theorem without proof.

Theorem 133 (accepted without proof) Given any polygonal region, any two complexes for that region have the same area sum.

If R is a polygonal region and C_1 and C_2 are two complexes for R, then the sum of the areas of the triangular regions of complex C_1 equals the sum of the areas of the triangular regions of complex C_2.

We are going to define the area of a polygonal region R to be the sum of the areas of the triangular regions of any complex C for R. When we compute the area, we get a positive real number as a result. The process of finding the area can be described as a function whose domain is the set of all polygonal regions and whose codomain is the set of non-negative real numbers. Here is a symbol that we can use for the set of all polygonal regions.

Definition 91 the set of all polygonal regions is denoted by $\mathcal{R}$.

We will use that symbol in the definition of area for polygonal regions.

Definition 92 the area function for polygonal regions
symbol: $Area$
spoken: the area function for polygonal regions
meaning: the function $Area: \mathcal{R} \rightarrow \mathbb{R}^{+}$ defined by

$$Area(R) = Area_{\blacktriangle}(\blacktriangle_1) + Area_{\blacktriangle}(\blacktriangle_2) + \cdots + Area_{\blacktriangle}(\blacktriangle_3) = \sum_{i=1}^{k} Area_{\blacktriangle}(\blacktriangle_i)$$

where $C = \{\blacktriangle_1, \blacktriangle_2, \dots, \blacktriangle_k\}$ is a complex for region R. (Theorem 133 guarantees that the resulting value does not depend on the choice of complex C.)

Now that we have a definition for the area of a polygonal region, we can restate some of the discussion from Sections 11.1 and 11.2 in a theorem.

Theorem 134 Properties of the Area Function for Polygonal Regions
Congruence: If R_1 and R_2 are triangular regions bounded by congruent triangles, then $Area(R_1) = Area(R_2)$.
Additivity: If R_1 and R_2 are polygonal regions whose interiors do not intersect, then $Area(R_1 \cup R_2) = Area(R_1) + Area(R_2)$.

In the next sections, we will discuss the area of Similar polygons.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 11.6 on page 258.

11.3. Using Area to Prove the Pythagorean Theorem

Here is a brief summary of key facts in our theory of area.

- The area of a triangle is $A = \frac{bh}{2}$, and it does not matter which side is chosen as the base. (from Definition 85 on page 244)
- The area of a polygonal region is obtained by subdividing the region into non-overlapping triangles and adding up the areas of those triangles. It does not matter which subdivision into non-overlapping triangles is used. (from Definition 92 on page 251)
- The Area Function for Polygonal Regions has the Congruence Property and the Additivity Property. (from Theorem 134, above)

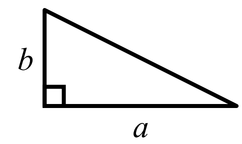
In the exercises for Section 11.2 (exercise found in Section 11.6 on page 258), you used those key facts to produce area formulas for a number of familiar shapes, including

- The area of a rectangle with length a and width b is $A = ab$.

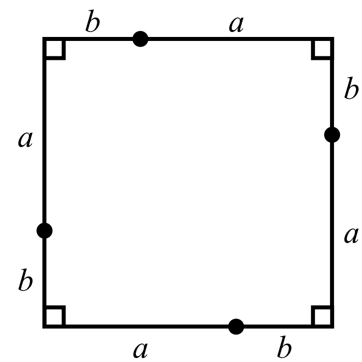
It turns out that the four bulleted facts above can be used to re-prove the Pythagorean Theorem. Here is one such proof:

Proof of the Pythagorean Theorem Using Area

Given a right triangle with legs of length a and b.



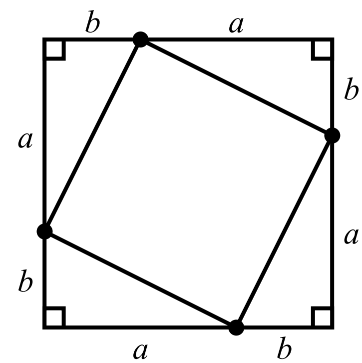
Introduce a square with sides of length $a + b$, with points on the sides of the square that divide each side of the square into segments of length a and b, as shown



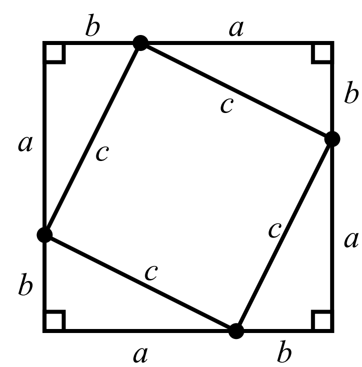
When the four points are joined, four right triangles are created.

The four triangles are congruent, and each is congruent to the original, given triangle.

(Question for the reader: How do we know that the four triangles are congruent?)

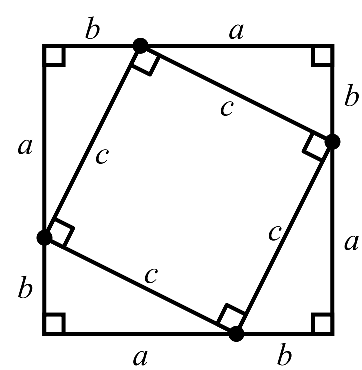


Therefore, the inner quadrilateral has sides of length c.



The quadrilateral on the inside has four sides of equal length, so it is certainly a rhombus. But in fact, the inner quadrilateral is also a square.

(Question for the reader: How do we know that the inner quadrilateral is actually a square?)



Now consider the following area calculations involving our final figure:

- The area of the large square is $(a+b)(a+b) = a^2 + 2ab + b^2$.
- The area of each triangle is $\frac{ab}{2}$.
- The area of the inner square is c^2.
- So the sum of the areas of the four triangles and the inner square is
$$4\left(\frac{ab}{2}\right) + c^2 = 2ab + c^2$$
- By additivity, the area of the large square must equal the sum of the areas of the four triangles and the inner square. That is, $a^2 + 2ab + b^2 = 2ab + c^2$
- Subtracting $2ab$ from both sides leaves us with the equation $a^2 + b^2 = c^2$. That is, the Pythagorean Theorem holds for the given right triangle.

End of Proof

This proof involving area seems to be conceptually much easier than our earlier proof of the Pythagorean Theorem, a proof that used only similarity. (Theorem 130 found in Section 10.3 Applications of Similarity on page 237) But keep in mind that many important details underly the simple proof involving area. For starters, the proof used the four bulleted facts at the start of this subsection. Those facts are highlights of more than fifteen pages of development in the current Chapter 11. And that development was possible only after the proof of the important theorem that says that for any triangle, the value of the quantity $\frac{bh}{2}$ does not depend on which side is chosen as the base, proven in Chapter 10. (Theorem 132, on page 238 of Section 10.3)

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 11.6 on page 258.

11.4. Areas of Similar Polygons

In Theorem 129 (About the ratios of lengths of certain line segments associated to similar triangles in Euclidean Geometry.) (found on page 236), we saw that for similar triangles, the ratio of the lengths of any pair of corresponding altitudes was the same as the ratio of the lengths of any pair of corresponding sides. That fact allows us to easily prove a very interesting result about the ratio of the areas of similar triangles

Theorem 135 about the ratio of the areas of similar triangles

The ratio of the areas of similar triangles is equal to the square of the ratio of the lengths of any pair of corresponding sides.

Proof

$$\begin{aligned}\frac{Area}{Area'} &= \frac{\frac{1}{2}base \cdot height}{\frac{1}{2}base' \cdot height'}\\ &= \frac{base \cdot height}{base' \cdot height'}\\ &= \left(\frac{base}{base'}\right) \cdot \left(\frac{height}{height'}\right)\end{aligned}$$

$$= \left(\frac{base}{base'}\right) \cdot \left(\frac{base}{base'}\right)$$
$$= \left(\frac{base}{base'}\right)^2$$

End of proof

It is reasonable to wonder if this fact generalizes to similar polygons. That is, is the ratio of the areas of a pair of similar polygons is equal to the square of the ratio of the lengths of any pair of corresponding sides? We will spend the rest of this section addressing that question. First, we should define similar polygons. You will notice that the definition reads a lot like our earlier Definition 80 of triangle similarity.

Definition 93 polygon similarity

To say that two polygons are *similar* means that there exists a correspondence between the vertices of the two polygons and the correspondence has these two properties:

- Each pair of corresponding angles is congruent.
- The ratios of the lengths of each pair of corresponding sides is the same.

If a correspondence between vertices of two polygons has the two properties, then the correspondence is called *a similarity.* That is, the expression *a similarity* refers to a particular correspondence of vertices that has the two properties.

The following statement has a straightforward proof. You will be asked to supply the proof in a homework exercise.

Theorem 136 polygon similarity is an equivalence relation

As with triangle similarity, there is subtlety in the notation used for similar polygons.

Definition 94 symbol for *a similarity* of two polygons

Symbol: $Polygon(P_1P_2 \ldots P_n) \sim Polygon(P_1'P_2' \ldots P_n')$.

Meaning: The correspondence $(P_1P_2 \ldots P_n) \leftrightarrow (P_1'P_2' \ldots P_n')$ of vertices is a similarity.

Now on to our question: is the ratio of the areas of a pair of similar polygons is equal to the square of the ratio of the lengths of any pair of corresponding sides?

We will start by attempting to answer the question for similar convex polygons. We will start with convex 3-gons, then consider convex 4-gons, convex 5-gons, etc. It will be helpful to first note the following algebraic fact about ratios:

$$If\ \frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_k}{b_k} = r\ then\ \frac{a_1 + a_2 + \cdots + a_k}{b_1 + b_2 + \cdots + b_k} = r$$

For convex 3-gons

Question: Is the ratio of the areas of a pair of similar convex 3-gons equal to the square of the ratio of the lengths of any pair of corresponding sides?

Answer: Yes, of course. A 3-gon is just a triangle, and every triangle is convex. Theorem 135 tells us that the ratio of the areas of a pair of similar triangles is equal to the square of the ratio of the lengths of any pair of corresponding sides.

For convex 4-gons

Question: Is the ratio of the areas of a pair of similar convex 4-gons equal to the square of the ratio of the lengths of any pair of corresponding sides?

Answer: Yes. Suppose that $Polygon(ABCD)$ and $Polygon(A'B'C'D')$ are convex and that $Polygon(ABCD) \sim Polygon(A'B'C'D')$. Suppose that the ratio of lengths of corresponding sides is $\frac{AB}{A'B'} = r$. Observe that $\Delta ABC \sim \Delta A'B'C'$ (by the *SAS* Similarity Theorem 128), so that $\frac{Area(\Delta ABC)}{Area(\Delta A'B'C')} = \left(\frac{AB}{A'B'}\right)^2 = r^2$ (by Theorem 135 about the ratio of the areas of similar triangles). Also observe that $\Delta BCD \sim \Delta B'C'D'$, so that $\frac{Area(\Delta BCD)}{Area(\Delta B'C'D')} = \left(\frac{BC}{B'C'}\right)^2 = \left(\frac{AB}{A'B'}\right)^2 = r^2$. Therefore, using the algebraic fact about ratios, we have $\frac{Area(\Delta ABC)+Area(\Delta BCD)}{Area(\Delta A'B'C')+Area(\Delta B'C'D')} = r^2$. Therefore, $\frac{Area(Polygon(ABCD))}{Area(Polygon(A'B'C'D'))} = r^2$.

For convex 5-gons

Question: Is the ratio of the areas of a pair of similar convex 5-gons equal to the square of the ratio of the lengths of any pair of corresponding sides?

Answer: Yes. Suppose that $Polygon(ABCDE)$ and $Polygon(A'B'C'D'E')$ are convex and that $Polygon(ABCDE) \sim Polygon(A'B'C'D'E')$. Suppose that the ratio of lengths of corresponding sides is $\frac{AB}{A'B'} = r$. Observe that $\Delta DEA \sim \Delta D'E'A'$ (by *SAS* Similarity), so that $\frac{Area(\Delta DEA)}{Area(\Delta D'E'A')} = \left(\frac{DE}{D'E'}\right)^2 = r^2$ (by Theorem 135).

The segments $\overline{DA}$ and $\overline{D'A'}$ partitioned each of the original 5-gons into a triangle and a quadrilateral. The fact that those two triangles are similar tells us that $\angle DAE \cong \angle D'A'E'$ and $\angle EAD \cong E'A'D'$ and that $\frac{DA}{D'A'} = r$. The fact about the congruent pairs of corresponding angles can be used to show that $\angle DAB \cong \angle D'A'B'$ and $\angle ADC \cong \angle A'D'C'$. In other words, the quadrilaterals $Polygon(ABCD)$ and $Polygon(A'B'C'D')$ are similar! Therefore, $\frac{Area(Polygon(ABCD))}{Area(Polygon(A'B'C'D'))} = r^2$ because of the fact that we proved earlier about the ratio of the areas of convex quadrilaterals. Therefore, using the algebraic fact about ratios, we have $\frac{Area(\Delta ABC)+Area(Polygon(ABCD))}{Area(\Delta A'B'C')+Area(Polygon(A'B'C'D'))} = r^2$. In other words, we have shown that $\frac{Area(Polygon(ABCDE))}{Area(Polygon(A'B'C'D'E'))} = r^2$.

For convex n-gons

Question: Is the ratio of the areas of a pair of similar convex n-gons equal to the square of the ratio of the lengths of any pair of corresponding sides?

Answer: Yes, but the proof structure is more sophisticated than the ones that we have seen so far in this book. Here is the general idea.

Suppose that $Polygon(P_1P_2 \dots P_n)$ and $Polygon(P_1'P_2' \dots P_n')$ are convex and that $Polygon(P_1P_2 \dots P_n) \sim Polygon(P_1'P_2' \dots P_n')$. Suppose that the ratio of lengths of corresponding sides is $\frac{AB}{A'B'} = r$. Observe that $\Delta P_{n-1}P_nP_1 \sim \Delta P_{n-1}'P_n'P_1'$ (by *SAS*

Similarity), so that $\frac{Area(\Delta P_{n-1}P_nP_1)}{Area(\Delta P_{n-1}'P_n'P_1')} = \left(\frac{P_{n-1}P_n}{P_{n-1}'P_n'}\right)^2 = r^2$ (by Theorem 135).

The segments $\overline{P_{n-1}'P_n'}$ and $\overline{P_{n-1}'P_n'}$ partitioned each of the original n-gons into a triangle and an n-1-gon. The fact that those two triangles are similar can be used to show $Polygon(P_1P_2 \dots P_{n-1})$ and $Polygon(P_1'P_2' \dots P_{n-1}')$ are similar. If we just knew that the equation $\frac{Area(Polygon(P_1P_2 \dots P_{n-1}))}{Area(Polygon(P_1'P_2' \dots P_{n-1}'))} = r^2$ was true, then the algebraic fact about ratios would tell us that $\frac{Area(\Delta P_{n-1}P_nP_1)+Area(Polygon(P_1P_2 \dots P_{n-1}))}{Area(\Delta P_{n-1}'P_n'P_1')+Area(Polygon(P_1'P_2' \dots P_{n-1}'))} = r^2$. In other words, we would know that $\frac{Area(Polygon(P_1P_2 \dots P_n))}{Area(Polygon(P_1'P_2' \dots P_n'))} = r^2$.
We can make this work with the method of Proof by Induction. That method will not be presented in this book, so we will not do the proof for convex *n*-gons in detail.

Having considered the question about the ratio of the areas of convex n-gons, an obvious question is: what if we drop the requirement that the *n*-gons be convex?

For general *n*-gons
Question: Is the ratio of the areas of a pair of similar *n*-gons (not necessarily convex) equal to the square of the ratio of the lengths of any pair of corresponding sides?
Answer: Yes, but the proof is beyond the level of this course.

We summarize the above discussion in the following theorem.

Theorem 137 about the ratio of the areas of similar *n*-gons
The ratio of the areas of a pair of similar *n*-gons (not necessarily convex) is equal to the square of the ratio of the lengths of any pair of corresponding sides.

In other words, if you double the lengths of the sides of a triangle, then you quadruple the area of the triangle. If you double the lengths of the sides of an *n*-gon, then you quadruple its area.

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 11.6 on page 258.

11.5. Area in High School Geometry Books

Now that we have finished our study of the theory of area in our Euclidean Geometry, it would be worthwhile to summarize what we have done and then compare our theory to the theory of area typically found in high school geometry books.

First, note that none of our eleven axioms says anything about area. Our whole theory was developed in definitions and theorems.

One of the two theorems that made our whole theory of area possible was in the previous chapter: Theorem 132 (In Euclidean Geometry, the product of $base \cdot height$ in a triangle does not depend on which side of the triangle is chosen as the base.) (found on page 238) That theorem is what enabled us to introduce the area function for triangular regions in Definition 85 (found in Section 11.1 on page 244).

Then we introduced the notion of a polygonal region and a complex (in Definition 88 on page 247 of Section 11.1).

A second theorem that made our whole theory of area possible was Theorem 133 ((accepted without proof) Given any polygonal region, any two complexes for that region have the same area sum.) (found in Section 11.2 on page 250) Although we officially accepted that theorem without proof, we studied most of the elements of the proof in a sequence of homework exercises for this chapter. That theorem is what enabled us to introduce the area function for polygonal regions (in Definition 92 on page 251 of Section 11.2). That is, the area of a polygonal region is obtained by subdividing the region into non-overlapping triangles and adding up the areas of those triangles. It does not matter which subdivision into non-overlapping triangles is used.

Finally, we proved that our area function has certain properties in Theorem 134 (Properties of the Area Function for Polygonal Regions) (found on page 251 of Section 11.2)

In summary, our theory of area is written in the precise mathematical language of functions, and it is developed without any additional axioms.

By contrast, high school books typically include axioms that simply declare that something called area exists. Here are the SMSG axioms having to do with area.

SMSG Postulate 17: To every polygonal region there corresponds a unique positive real number called its area.
SMSG Postulate 18: If two triangles are congruent, then the triangular regions have the same area.
SMSG Postulate 19: Suppose that the region R is the union of two regions R_1 and R_2. If R_1 and R_2 intersect at most in a finite number of segments and points, then the area of R is the sum of the areas of R_1 and R_2.
SMSG Postulate 20: The area of a rectangle is the product of the length of its base and the length of its altitude.

Notice that the SMSG Postulates do not use the terminology of functions. Furthermore, the SMSG postulates simply declare that the area has certain properties, the same properties that we were able to prove in theorems.

It is worth noting an important consequence of the SMSG approach to area. Recall our discussion of the Proof of the Pythagorean Theorem Using Area, at the end of Section 11.3. (That section starts on page 251.) We observed that in our book, there is the following sequence of developments.

(1) The concept of Similarity is developed in Chapter 10.
(2) The Pythagorean Theorem is proven with a moderately-difficult proof involving similarity. (Theorem 130 found in Section 10.3 Applications of Similarity on page 237)
(3) The theory of Area is developed in Chapter 11.
(4) The Pythagorean Theorem is re-proven with a very simple proof using Area (in Chapter 11).

We discussed the fact that although the proof of the Pythagorean Theorem using area looked very simple, it did involve a lot of underlying theory. So the proof involving Similarity was simpler overall, even though it looked more difficult than the proof involving area.

But in a book that uses the SMSG Postulates, one is simply given the theory of Area in the axioms; no development of Area theory is required. If that is the approach, then the smartest way to prove the Pythagorean Theorem would be to just use area, taking advantage of the SMSG axioms about area. In such a book, the area-based proof really would be concepually simpler than a proof that used similarity.

11.6. Exercises for Chapter 11

Exercises for Section 11.2 (The Area of a Polygonal Region) (Section starts on page 250.)

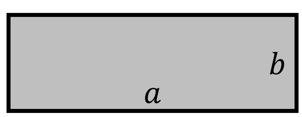
In our definition of the area of a polygonal region, the only regions whose areas are computed by a formula are the triangular regions. The area of any other kind of polygonal region is obtained by first identifying a complex for the region, and then finding the area of the triangles in the complex. But if we stick to this plan for computing area, we will quickly see some familiar formulas emerge for the area of certain kinds of polygonal regions. Our first exercises explore this.

[1] Find the formula for the area of each shape by identifying a complex and then finding the sum of the triangular regions in the complex using the formula for the area of a triangular region,

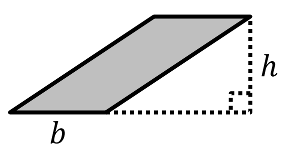
$$Area_{\blacktriangle}(\blacktriangle ABC) = \frac{bh}{2}$$

Provide large drawings showing your triangulations clearly.

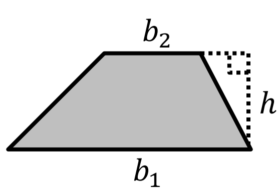
(a) Rectangle



(b) Parallelogram

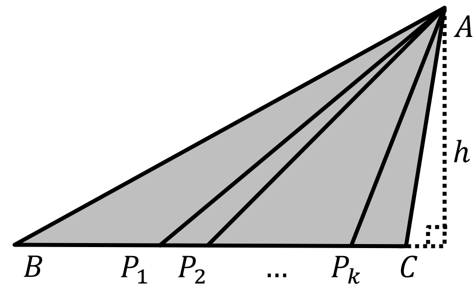


(c) Trapezoid

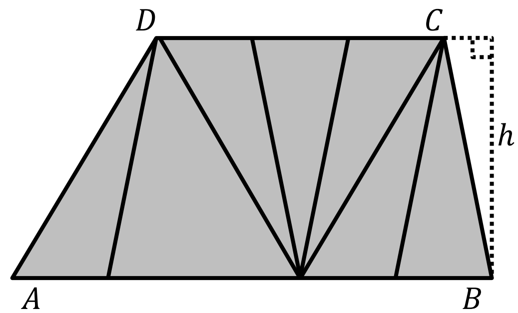


Theorem 133 (found on page 250) states that given any polygonal region, any two complexes for that region have the same area sum. We accepted this theorem without proof, but we should have some sense of how the proof of the theorem would work. The next few exercises show how some pieces of the proof would work.

[2] Show that for any triangle ΔABC, if $P_1, P_2, \ldots, P_k$ are points on side $\overline{BC}$, then the complex of $k+1$ triangles $\{\blacktriangle BP_1A, \blacktriangle P_1P_2A, \ldots, \blacktriangle P_kCA\}$ has an area sum that is equal to the area of the complex $\{\blacktriangle ABC\}$.



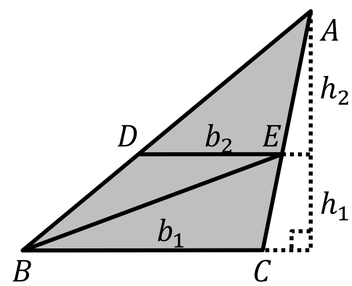
[3] Show that for any trapezoid $ABCD$, if a complex of triangles is formed by connecting points on the upper and lower bases, then the complex has an area sum that is equal to the area given by the formula that you found in problem [3].



[4] Triangle ΔABC is split by a segment $\overline{DE}$ parallel to side $\overline{BC}$. Show that the following two numbers are the same:

(1) The area of complex $\{\blacktriangle ABC\}$.

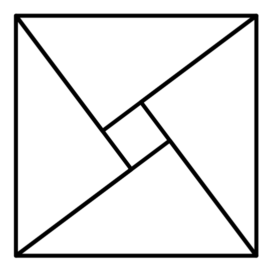
(2) The area of complex $\{\blacktriangle ADE, \blacktriangle DBE, \blacktriangle BCE\}$.



Exercises for Section 11.3 (Using Area to Prove the Pythagorean Theorem) (page 251)

[5] In the Proof of the Pythagorean Theorem Using Area, which starts on page 251 in Section 11.3, there are two questions for the reader. Answer those questions.

[6] Use the diagram at right as the inspiration for a proof of the Pythagorean Theorem using area.
Hint: Use as your model the proof that starts on page 251 in Section 11.3. That is, provide a sequence of drawings and the same amount of explanation. (But don't leave any unanswered questions for the reader!)



[7] Come up with another proof of the Pythagorean Theorem using area, not like the two presented in this chapter.
Hint: You can start by doing a web search for "proof of the pythagorean theorem", and in the search results, choose "images". This should get you a large collection of images that you can use for inspiration, and will also get you some fully-explained proofs. But you will need to write a proof in your own words, in full detail, using a sequence of drawings, not just one drawing.

Exercises for Section 11.4 (Areas of Similar Polygons) (Section starts on page 253.)

[8] Justify the steps in the proof of Theorem 135 (about the ratio of the areas of similar triangles) (found on page 253).

[9] Prove Theorem 136 (polygon similarity is an equivalence relation) (found on page 254).

[10] Prove the algebraic fact about ratios cited in Section 11.4:

$$If\,\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_k}{b_k}=r\;then\;\frac{a_1+a_2+\cdots+a_k}{b_1+b_2+\cdots+b_k}=r$$

Hint: Notice that $a_1 = rb_1$ and $a_2 = rb_2$, etc. Use this to rewrite the $a_1 + a_2 + \cdots + a_k$ in the numerator.

[11] Make drawings to illustrate the discussion preceeding Theorem 137 (about the ratio of the areas of similar n-gons) (found on page 256). That is, draw $Polygon(ABCD)$ and $Polygon(A'B'C'D')$ that are convex and similar, and illustrate the steps in the discussion about convex 4-gons. Then draw $Polygon(ABCDE)$ and $Polygon(A'B'C'D'E')$ that are convex and similar, and illustrate the steps in the discussion about convex 5-gons. Then draw $Polygon(P_1P_2 \dots P_n)$ and $Polygon(P_1'P_2' \dots P_n')$ that are convex and similar, and illustrate the discussion about convex n-gons.

[12] If you want to triple the area of a square, by what factor should you multiply the lengths of the sides?

[13] (A) Suppose that a 30-60-90 triangle has a hypotenuse that is x units long. How long is the short leg of the triangle? (Hint: Theorem 62, found on page 172, tells us that the short leg will be opposite the angle of measure 30. Consider first an equilateral triangle whose sides are x units long. Draw the altitude from one vertex to the opposite side. Theorem 85, found on page 199, can be used to say something about the lengths of the segments created on the opposite side.)

(B) Suppose that a 30-60-90 triangle has a hypotenuse that is x units long. How long is the long leg of the triangle? (Hint: Use the answer to (A) and the Pythagorean Theorem.

(C) Using your answers to (A) and (B), find the area of a 30-60-90 triangle.

(D) Find the area of an equilateral triangle whose sides are x units long.

[14] Suppose that ΔABC is a right triangle with right angle at C. Let $a = BC$ and $b = CA$ and $c = AB$. The Pythagorean Theorem says that $a^2 + b^2 = c^2$. This can be interpreted geometrically: Suppose that squares are constructed on each side of the triangle. Then those squares will have areas a^2 and b^2 and c^2. The Pythagorean Theorem says that the sum of the areas of the two smaller squares equals the are of the larger square.

There is a generalization of this idea to other shapes. For instance, instead of constructing squares on each side of triangle ΔABC, construct three triangles that are similar to each other. That is, suppose that D, E, F are three points such that $\Delta ABD \sim \Delta BCE \sim \Delta CAF$. Show that

$$Area(\Delta BCE) + Area(\Delta CAF) = Area(\Delta ABD)$$

Hint: Divide both sides of the Pythagorean Theorem by c^2 to obtain the new equation

$$\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{c^2}{c^2}$$

That is,

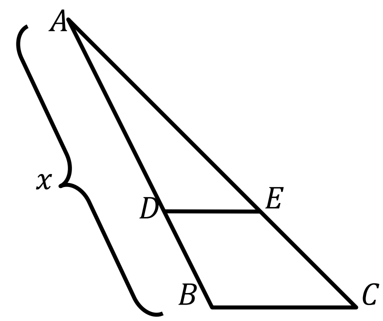
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Then use Theorem 135, found on page 253, to rewrite this equation in terms of ratios of areas of similar triangles. Then rearrange the equation to get the form asked for in the problem statement.

[15] Refer to the drawing at right, which is not drawn to scale.

$$AB = x$$
$$\overleftrightarrow{BC} \| \overleftrightarrow{DE}$$
$$Area(\Delta ABC) = 3Area(\Delta ADE)$$

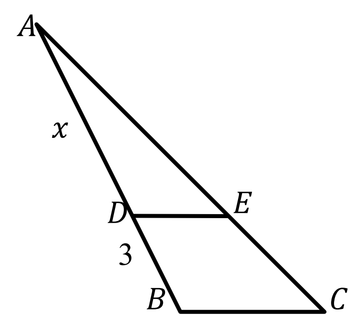
Find AD in terms of x.



[16] Refer to the drawing at right, which is not drawn to scale.

$$AD = x$$
$$DB = 3$$
$$\overleftrightarrow{BC} \| \overleftrightarrow{DE}$$
$$Area(\Delta ABC) = 16Area(\Delta ADE)$$

Find x. Show your work.

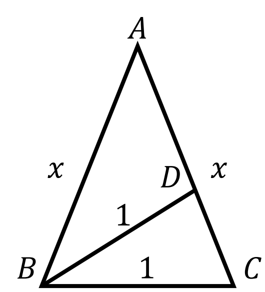


[17] In the figure at right,

$$AB = AC = x$$
$$BC = BD = 1$$

Let y be the value of the ratio of the areas:

$$y = \frac{Area(\Delta ABC)}{Area(\Delta BCD)}$$



Find y in terms of x. Show all steps that lead to your answer.

Hint: Each triangle has two congruent sides. Cite a theorem to identify congruent angles. Then identify two similar triangles. (Draw them side-by-side with the same orientation.)

[18] A regular hexagon called hex_1 has sides of length x. A second hexagon called hex_2 is created by joining the midpoints of the sides of hex_1. Let y be the value of the ratio of the areas:

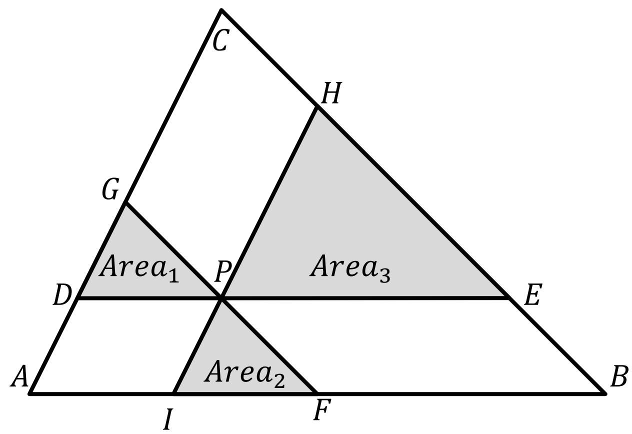
$$y = \frac{Area(hex_1)}{Area(hex_2)}$$

Find y in terms of x. Show all steps that lead to your answer.

[19] In the figure at right,

$$\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$$
$$\overleftrightarrow{BC} \parallel \overleftrightarrow{FG}$$
$$\overleftrightarrow{CA} \parallel \overleftrightarrow{HI}$$

The goal is to find a relationship between $area(\Delta ABC)$ and $Area_1$ and $Area_2$ and $Area_3$.



(A) Prove that $\Delta ABC \sim \Delta DPG \sim \Delta IFP \sim \Delta PEH$.

Define symbols

- Let $x_1 = length\ of\ base\ of\ \Delta DPG = length(\overline{PD})$.
- Let $x_2 = length\ of\ base\ of\ \Delta IFP = length(\overline{IF})$.
- Let $x_3 = length\ of\ base\ of\ \Delta PEH = length(\overline{PE})$.
- Let $x = length\ of\ base\ of\ \Delta ABC = length(\overline{AB})$.

In the next three questions, you will apply Theorem 135 to get ratios of areas in terms of the symbols x_1, x_2, x_3, x.

(B) Find the ratio $\frac{Area(\Delta DPG)}{Area(\Delta ABC)}$.
(C) Find the ratio $\frac{Area(\Delta IFP)}{Area(\Delta ABC)}$.
(D) Find the ratio $\frac{Area(\Delta PEH)}{Area(\Delta ABC)}$.

The next four questions are the main computation.

(E) Get an equation expressing x in terms of x_1, x_2, x_3.
(F) Divide both sides of this equation by x.
(G) Replace these expressions with square roots of the ratios of areas from questions (B),(C),(D).
(H) Finally, multiply both sides of this equation by $\sqrt{area(\Delta ABC)}$. The result should be an equation that expresses a relationship between $area(\Delta ABC)$ and $Area_1$ and $Area_2$ and $Area_3$.

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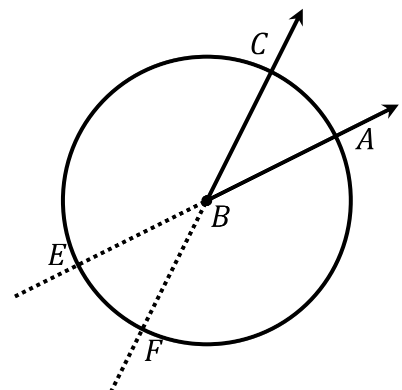
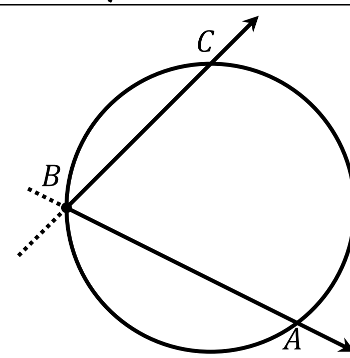
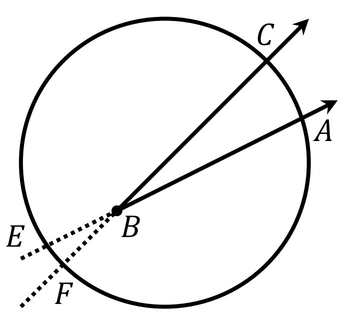
12. Euclidean Geometry IV: Circles

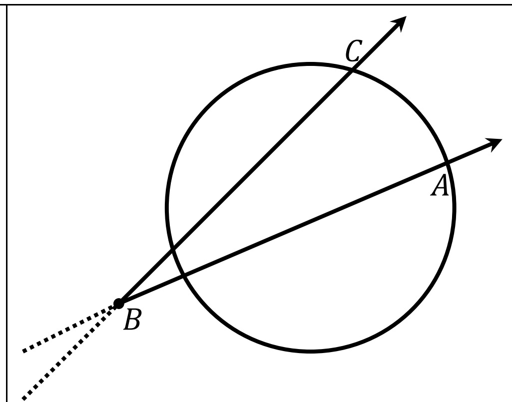
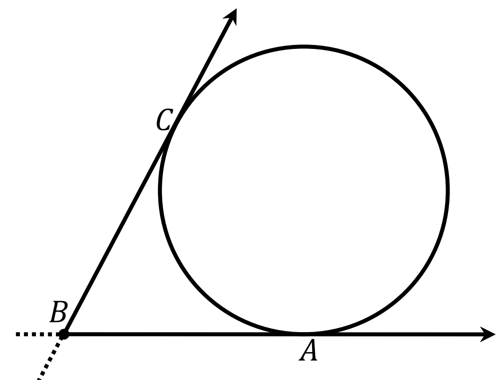
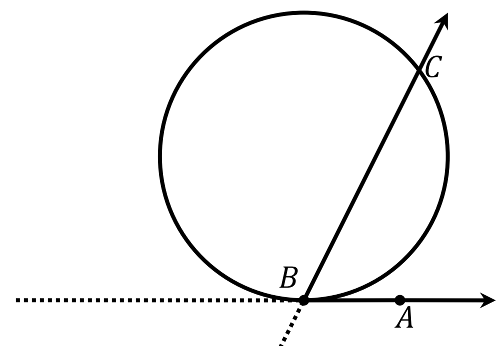
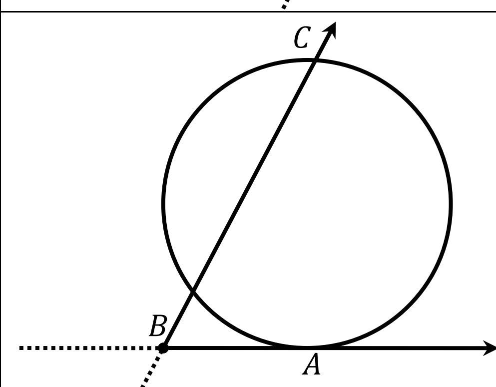
We previously studied circles in Chapter 8, in the context of Neutral Geometry. All of the theorems of that chapter are also theorems that are true about circles in Euclidean Geometry. In this chapter we will study some results that are strictly Euclidean. That is, their proofs require the use of the Euclidean Parallel Axiom <EPA>, and their statements are not true in Neutral Geometry. We will study angles that intersect circles. Arcs will be introduced, and we will study the relationships between the measures of angles and the arcs that they intercept.

12.1. Circular Arcs

In this chapter, we will be interested in angles that intersect circles but only in seven particular configurations. Here are the seven types, presented as a single definition.

Definition 95 seven types of angles intersecting circles

Type 1 Angle (*Central Angle*) A *central angle* of a circle is an angle whose rays lie on two secant lines that intersect at the center of the circle. In the picture at right, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CF}$ are secant lines that intersect at the center point B of the circle. Angle $\angle ABC$ is a central angle. So are angles $\angle CBE, \angle EBF, \angle FBA$.	
Type 2 Angle (*Inscribed Angle*) An *inscribed angle* of a circle is an angle whose rays lie on two secant lines that intersect on the circle and such that each ray of the angle intersects the circle at one other point. In other words, an angle of the form $\angle ABC$, where A, B, C are three points on the circle. In the picture at right, angle $\angle ABC$ is an inscribed angle.	
Type 3 Angle Our third type of an angle intersecting a circle is an angle whose rays lie on two secant lines that intersect at a point that is inside the circle but is not the center of the circle. In the picture at right, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CF}$ are secant lines that intersect at point B in the interior of the circle. Angle $\angle ABC$ is an angle of type three. So are angles $\angle CBE, \angle EBF, \angle FBA$.	

Type 4 Angle Our fourth type of an angle intersecting a circle is an angle whose rays lie on two secant lines that intersect at a point that is outside the circle and such that each ray of the angle intersects the circle. In the picture at right, angle $\angle ABC$ is an angle of type four.	
Type 5 Angle Our fifth type of an angle intersecting a circle is an angle whose rays lie on two tangent lines and such that each ray of the angle intersects the circle. Because the rays lie in tangent lines, we know that each ray intersects the circle exactly once. In the picture at right, angle $\angle ABC$ is an angle of type five.	
Type 6 Angle Our sixth type of an angle intersecting a circle is an angle whose vertex lies on the circle and such that one ray contains a chord of the circle and the other ray lies in a line that is tangent to the circle. In the picture at right, angle $\angle ABC$ is an angle of type six.	
Type 7 Angle Our seventh type of an angle intersecting a circle is an angle whose rays lie on a secant line and tangent line that intersect outside the circle and such that each ray of the angle intersects the circle. In the picture at right, angle $\angle ABC$ is an angle of type seven.	

Notice that in each of the seven pictures above, there is a portion of the circle that lies in the interior of the angle. Those subsets can be described using the terminology of circular arcs. That terminology is the subject of the next section.

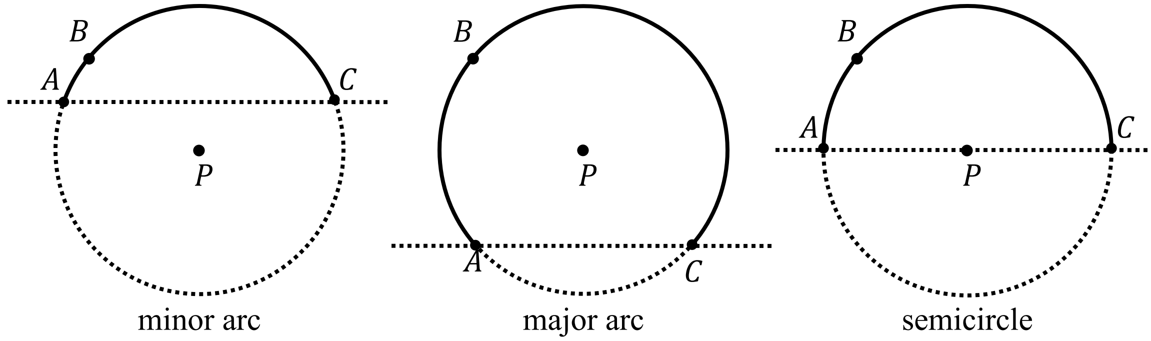
Recall that in Euclidean Geometry, any three non-collinear points A, B, C lie on exactly one circle. (by Theorem 107, found on page 215) We could use the symbol $Circle(A, B, C)$ to denote the unique circle that passes through those three points. Also recall that the Axiom of Separation gives us the notion of the two half-planes determined by a line (Definition 17, found on page 61). We can use the symbol H_B to denote the half-plane determined by line $\overleftrightarrow{AC}$ that contains point B. We will use the terminology of half-planes in our definition of circular arcs.

Definition 96 Circular Arc

Symbol: $\widehat{ABC}$
Spoken: arc A, B, C
Usage: A, B, C are non-collinear points.
Meaning: the set consisting of points A and C and all points of $Circle(A, B, C)$ that lie on the same side of line $\overleftrightarrow{AC}$ as point B.
Meaning in Symbols: $\widehat{ABC} = \{A \cup C \cup (Circle(A, B, C) \cap H_B)\}$
Additional terminology:

- Points A and C are called the endpoints of arc $\widehat{ABC}$.
- The interior of the arc is the set $Circle(A, B, C) \cap H_B$.
- If the center P lies on the opposite side of line $\overleftrightarrow{AC}$ from point B, then arc $\widehat{ABC}$ is called a minor arc.
- If the center P of $Circle(A, B, C)$ lies on the same side of line $\overleftrightarrow{AC}$ as point B,then arc $\widehat{ABC}$ is called a major arc.
- If the center P lies on line $\overleftrightarrow{AC}$, then arc $\widehat{ABC}$ is called a semicircle.

Picture:



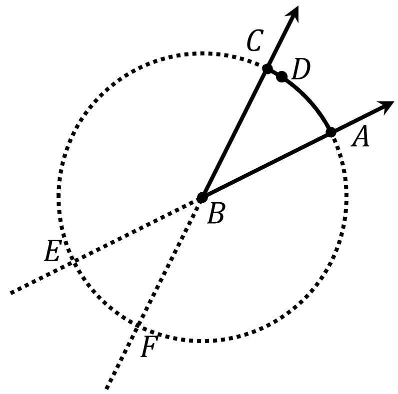
Now that we have the terminology of circular arcs, we can resume the discussion that we started above about the portion of the circle that lies in the interiors of the seven types of angles. The termology of an angle intercepting an arc will be useful.

Definition 97 angle intercepting an arc

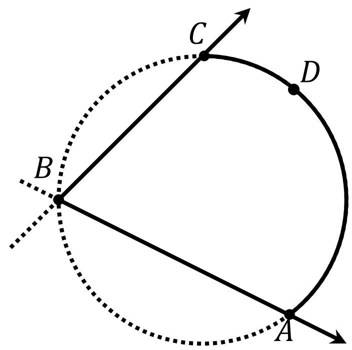
We say that an angle intercepts an arc if each ray of the angle contains at least one endpoint of the arc and if the interior of the arc lies in the interior of the angle.

There is a bit of subtlety in the way that this definition is written. It is worthwhile to go examine our seven types of angles intersecting circles, and consider the arcs that each type intersects.

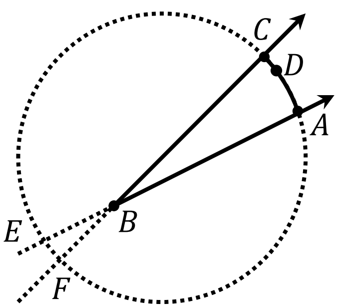
Type 1 Central angle $\angle ABC$ intercepts arc $\widehat{ADC}$.



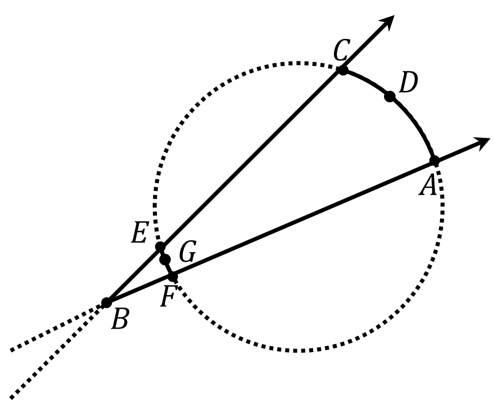
Type 2 Inscribed angle $\angle ABC$ intercepts arc $\widehat{ADC}$.



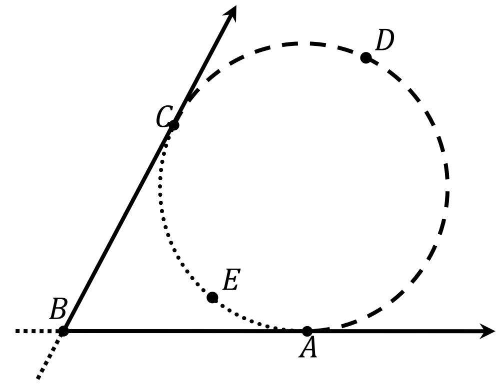
Type 3 angle $\angle ABC$ intercepts arc $\widehat{ADC}$.



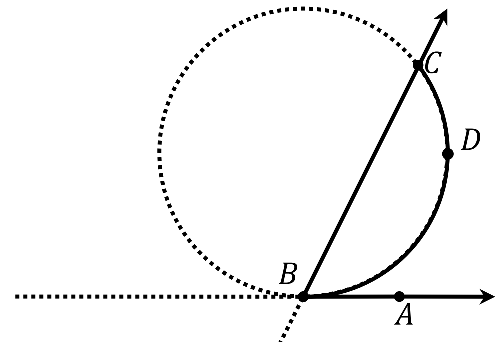
Type 4 angle $\angle ABC$ intercepts arc $\widehat{ADC}$ and also intercepts arc $\widehat{EGF}$. Notice that the angle intercepts two arcs. This is allowed by the definition.



Type 5 angle $\angle ABC$ intercepts dashed arc $\widehat{ADC}$ and also intercepts dotted arc $\widehat{AEC}$. Notice that the angle intercepts two arcs and the arcs share both of their endpoints.

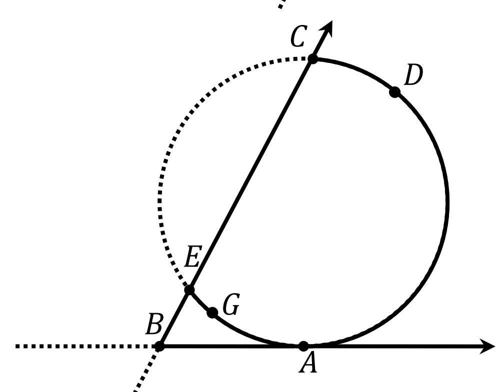


Type 6 angle $\angle ABC$ intercepts arc $\widehat{BDC}$. Notice that the angle only intercepts one arc. But both endpoints of the arc lie on ray $\overrightarrow{BC}$, and one endpoint of the arc lies one ray $\overrightarrow{BA}$. This is allowed by the definition, because each ray of the angle contains at least one endpoint of the arc.



Type 7 angle $\angle ABC$ intercepts arc $\widehat{ADC}$ and also intercepts arc $\widehat{AEG}$. Notice that the angle intercepts two arcs and the arcs share an endpoint.

Question for the reader: Why does angle $\angle ABC$ *not* intercept arc $\widehat{EAC}$?



Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 12.6 on page 281..

12.2. Angle Measure of an Arc

For a given arc on a circle, we would like to have some way of quantifying how far around the circle the arc goes. That leads us to the idea of the angle measure of an arc. We want to state a definition that uses function notation, so it will help for us to have a symbol for the set of all circular arcs.

Definition 98 the symbol for the set of all circular arcs is $\hat{\mathcal{A}}$.

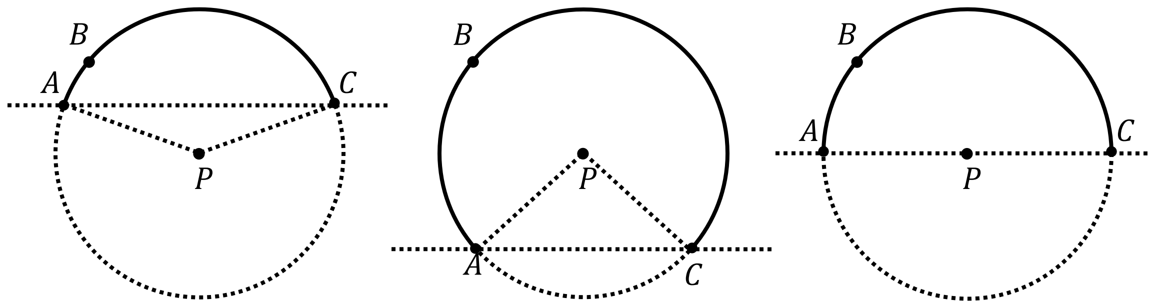
We will define the angle measure of an arc using a function.

Definition 99 the angle measure of an arc
Symbol: $\widehat{m}$

Name: the Arc Angle Measurement Function
Meaning: The function $\hat{m}: \hat{\mathcal{A}} \to (0{,}360)$, defined in the following way:

- If $\widehat{ABC}$ is a minor arc, then $\hat{m}(\widehat{ABC}) = m(\angle APC)$, where point P is the center of the circle.
- If $\widehat{ABC}$ is a major arc, then $\hat{m}(\widehat{ABC}) = 360 - m(\angle APC)$, where point P is the center of the circle.
- If $\widehat{ABC}$ is a semicircle, then $\hat{m}(\widehat{ABC}) = 180$.

Picture:



minor arc $\hat{m}(\widehat{ABC}) = m(\angle APC)$ | major arc $\hat{m}(\widehat{ABC}) = 360 - m(\angle APC)$ | semicircle $\hat{m}(\widehat{ABC}) = 180$

We see that a very small minor arc will have an arc angle measure near zero, while a major arc that goes almost all the way around the circle will have an arc angle measure near 360. And we see why, in this definition, the measure of an arc is always greater than zero and less than 360.

Our first theorem is an easy corollary of the definition of arc angle measurement.

Theorem 138 If two distinct arcs share both endpoints, then the sum of their arc angle measures is 360. That is, if $\widehat{ABC}$ and $\widehat{ADC}$ are distinct, then $\hat{m}(\widehat{ABC}) + \hat{m}(\widehat{ADC}) = 360$.

Our second theorem uses only the definition of arc angle measurement and has a very simple proof. You will be asked to prove it in a homework exercise.

Theorem 139 Two chords of a circle are congruent if and only if their corresponding arcs have the same measure.

Recall the Angle Measure Addition Axiom from the Axioms for Neutral Geometry (Definition 17, found on page 61)

<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Because of the way that arc angle measure is defined in terms of angle measure, it should be no surprise that arc angle measure will behave in an analogous way. Here is the theorem and its proof.

Theorem 140 The Arc Measure Addition Theorem
If $\widehat{ABC}$ and $\widehat{CDE}$ are arcs that only intersect at C, then $\hat{m}(\widehat{ACE}) = \hat{m}(\widehat{ABC}) + \hat{m}(\widehat{CDE})$.

The proof is a rather obnoxious one involving five cases. In a homework exercise, you will be asked to provide drawings.

Proof (for readers interested in advanced topics and for graduate students)

Suppose that $\widehat{ABC}$ and $\widehat{CDE}$ and $\widehat{ACE}$ are arcs.

There are exactly five possibilities:

(1) $\widehat{ABC}$ and $\widehat{CDE}$ and $\widehat{ACE}$ are minor arcs.

(2) $\widehat{ABC}$ and $\widehat{CDE}$ are minor arcs, but $\widehat{ACE}$ is a semicircle.

(3) $\widehat{ABC}$ and $\widehat{CDE}$ are minor arcs, but $\widehat{ACE}$ is a major arc.

(4) $\widehat{ABC}$ is a minor arc, but $\widehat{CDE}$ is a semicircle, and $\widehat{ACE}$ is a major arc.

(5): $\widehat{ABC}$ is a minor arc, but $\widehat{CDE}$ and $\widehat{ACE}$ are major arcs.

We must show that $\widehat{m}(\widehat{ACE}) = \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE})$ in every case.

Case 1:

Suppose that $\widehat{ABC}$ and $\widehat{CDE}$ and $\widehat{ACE}$ are minor arcs. **(Make a drawing)**

Then

$$\begin{aligned}\widehat{m}(\widehat{ACE}) &= m(\angle APE) \quad (Justify)\\ &= m(\angle APC) + m(\angle CPE) \quad (Justify)\\ &= \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE}) \quad (Justify)\end{aligned}$$

So $\widehat{m}(\widehat{ACE}) = \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE})$ in this case.

Case 2:

Suppose that $\widehat{ABC}$ and $\widehat{CDE}$ are minor arcs, but $\widehat{ACE}$ is a semicircle. **(Make a drawing.)**

Then

$$\begin{aligned}\widehat{m}(\widehat{ACE}) &= 180 \quad (Justify)\\ &= m(\angle APC) + m(\angle CPE) \quad (Justify)\\ &= \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE}) \quad (Justify)\end{aligned}$$

So $\widehat{m}(\widehat{ACE}) = \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE})$ in this case.

Case 3:

Suppose that $\widehat{ABC}$ and $\widehat{CDE}$ are minor arcs, but $\widehat{ACE}$ is a major arc. **(Make a drawing.) (You provide the steps to prove this case.)**

Case 4:

Suppose that $\widehat{ABC}$ is a minor arc, but $\widehat{CDE}$ is a semicircle, and $\widehat{ACE}$ is a major arc. **(Make a drawing.) (You provide the steps to prove this case.)**

Case 5:

Suppose that $\widehat{ABC}$ is a minor arc, but $\widehat{CDE}$ and $\widehat{ACE}$ are major arcs. **(Make a drawing.) (You provide the steps to prove this case.)**

Conclusion:

We see that $\widehat{m}(\widehat{ACE}) = \widehat{m}(\widehat{ABC}) + \widehat{m}(\widehat{CDE})$ in every case.

End of Proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 12.6 on page 281.

12.3. The Measure of Angles Related to the Measures of Arcs that they Intercept

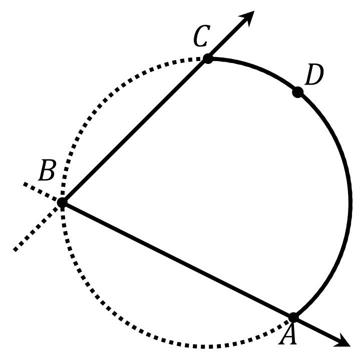
In Section 12.1, we discussed angles intersecting circles, identifying seven types of angles that would be of interest to us. We also defined circular arcs and the notion of an angle intercepting an arc. In Section 12.2, we defined the angle measure of an arc. In this section, we will study the relationships between the measures of angles that intersect circles and the measures of the arcs that they intercept. We will consider all seven types of angles from Section 12.1.

We already know that the angle measure of a central angle (Type 1) is equal to the arc angle measure of the arc intercepted by the angle, because that is how the arc angle measure was defined.

Next, we will consider inscribed angles (Type 2). Here is a theorem presenting the relationship. You will justify the proof steps in a homework exercise.

Theorem 141 the angle measure of an inscribed angle (Type 2) is equal to half the arc angle measure of the intercepted arc.

Type 2 Inscribed angle $\angle ABC$ intercepts arc $\widehat{ADC}$.



Proof

(1) Suppose that an inscribed angle is given. There are three possibilities
- (i) The center of the circle lies on one of the rays of the angle.
- (ii) The center of the circle lies in the interior of the angle.
- (ii) The center of the circle lies in the exterior of the angle.

Case (i)

(2) Suppose that the center of the circle lies on one of the rays of the angle. **(Make a drawing.)**

(3) (Labeling) Label the angle $\angle ABC$ so that the center P lies on ray $\overrightarrow{BA}$. Let D be some point in the interior of the arc intercepted by $\angle ABC$, so that the arc can be denoted $\widehat{ADC}$. Let $x = \widehat{m}\left(\widehat{ADC}\right)$ and let $y = m(\angle ABC)$. **(Update your drawing.)** Our goal is to find y in terms of x.

(4) $m(\angle APC) = x$. **(Justify. Update your drawing.)**

(5) $m(\angle APC) = m(\angle PBC) + m(\angle PCB)$. **(Justify)**

(6) $\angle PCB \cong \angle PBC$. **(Justify)**

(7) Therefore, $m(\angle PCB) = y$. **(Update your drawing.)**

(8) $x = y + y$. (Substituted from steps 3 and 7 into step 5.)

(9) Therefore, $y = \frac{x}{2}$. So in this case, we see that the angle measure of the inscribed angle is equal to half the arc angle measure of the intercepted arc.

Case (ii)

(10) Suppose that the center of the circle lies in the interior of the angle. **(Make a new drawing.)**

(11) (Labeling) Label the angle $\angle ABC$ and the center P. Let point E be at the other end of a diameter from point B. Let D be some point in the interior of the arc intercepted by $\angle ABE$, so that the arc can be denoted $\widehat{ADE}$. Let F be some point in the interior of the arc intercepted by $\angle EBC$, so that the arc can be denoted $\widehat{EFC}$. **(Update your drawing.)** Our goal is to show that $m(\angle ABC) = \frac{\widehat{m}(\widehat{AEC})}{2}$.

(12) Then

$$\begin{aligned} m(\angle ABC) &= m(\angle ABE) + m(\angle EBC) \quad (\boldsymbol{Justify.}) \\ &= \frac{\widehat{m}(\widehat{ADE})}{2} + \frac{\widehat{m}(\widehat{EFC})}{2} \quad (\boldsymbol{Justify.}) \\ &= \frac{\widehat{m}(\widehat{ADE}) + \widehat{m}(\widehat{EFC})}{2} \\ &= \frac{\widehat{m}(\widehat{AEC})}{2} \quad (\boldsymbol{Justify.}) \end{aligned}$$

So in this case, we see that the angle measure of the inscribed angle is equal to half the arc angle measure of the intercepted arc.

Case (iii)

(13) Suppose that the center of the circle lies in the exterior of the angle. **(Make a new drawing.)**

(14) (Labeling) Label the angle $\angle ABC$ and the center P so that P and C lie on opposite sides of line $\overleftrightarrow{AB}$. Let point F be at the other end of a diameter from point B. Let E be some point in the interior of the arc intercepted by $\angle ABF$, so that the arc can be denoted $\widehat{AEF}$. Let D be some point in the interior of the arc intercepted by $\angle ABC$, so that the arc can be denoted $\widehat{ADC}$. **(Update your drawing.)** Our goal is to show that $m(\angle ABC) = \frac{\widehat{m}(\widehat{ADC})}{2}$.

(15) Then

$$\begin{aligned} m(\angle ABC) &= m(\angle FBC) - m(\angle FBA) \quad (Justify.) \\ &= \frac{\widehat{m}(\widehat{FAC})}{2} - \frac{\widehat{m}(\widehat{FEA})}{2} \quad (Justify.) \\ &= \frac{\widehat{m}(\widehat{FAC}) - \widehat{m}(\widehat{FEA})}{2} \\ &= \frac{\widehat{m}(\widehat{ADC})}{2} \quad (Justify.) \end{aligned}$$

So in this case, we see that the angle measure of the inscribed angle is equal to half the arc angle measure of the intercepted arc.

Conclusion

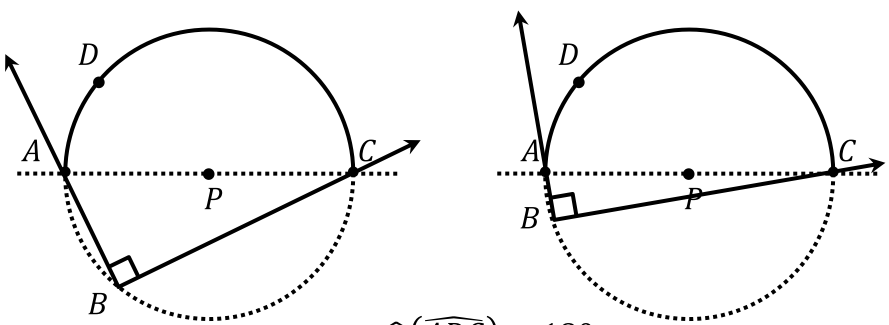
(16) We see that in every case, the angle measure of the inscribed angle is equal to half the arc angle measure of the intercepted arc.

End of Proof

There is a very simple corollary that turnes out to be extremely useful:

Theorem 142 (Corollary) Any inscribed angle that intercepts a semicircle is a right angle.

The two figures below illustrate Theorem 142



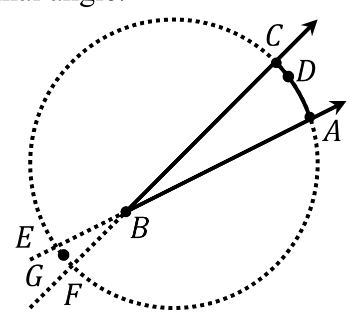
$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC})}{2} = \frac{180}{2} = 90$$

Now on to Type 3 angles. You will justify the steps of the proof in a homework exercise.

Theorem 143 the angle measure of an angle of Type 3.
The angle measure of an angle of Type 3 is equal to the average of the arc angle measures of two arcs. One arc is the arc intercepted by the angle, itself. The other arc is the arc intercepted by the angle formed by the opposite rays of the original angle.

In the figure at right, Type 3 angle $\angle ABC$ intercepts arc $\widehat{ADC}$. The theorem states that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) + \hat{m}(\widehat{EGF})}{2}$$



Proof

(1) Suppose that an angle of Type 3 is given. **(Make a drawing.)**

(2) (Labeling) Label the angle $\angle ABC$. Ray $\overrightarrow{BA}$ lies in a secant line. Let E be the second point of intersection of that secant line and the circle. Ray $\overrightarrow{BC}$ lies in a secant line. Let F be the second point of intersection of that secant line and the circle. Let D be some point in the interior of the arc intercepted by angle $\angle ABC$, so that the arc can be denoted $\widehat{ADC}$. Let G be some point in the interior of the arc intercepted by angle $\angle EBF$, so that the arc can be denoted $\widehat{EGF}$. **(Update your drawing.)** Our goal is to show that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) + \hat{m}(\widehat{EGF})}{2}$$

(3) Then

$$\begin{aligned} m(\angle ABC) &= m(\angle BEC) + m(\angle BCE) \quad (Justify.) \\ &= \frac{\hat{m}(\widehat{ADC})}{2} + \frac{\hat{m}(\widehat{EGF})}{2} \quad (Justify.) \\ &= \frac{\hat{m}(\widehat{ADC}) + \hat{m}(\widehat{EGF})}{2} \end{aligned}$$

End of proof

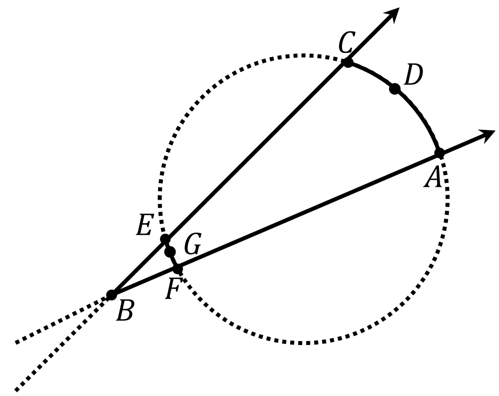
Now on to Type 4 angles. You will prove the following theorem in a homework exercise.

Theorem 144 the angle measure of an angle of Type 4.
The angle measure of an angle of Type 4 is equal to one half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.)

In the figure at right, Type 4 angle $\angle ABC$ intercepts arc $\widehat{ADC}$ and also intercepts arc $\widehat{EGF}$. Notice that the angle intercepts two arcs. This is allowed by the definition. The theorem states that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) - \hat{m}(\widehat{EGF})}{2}$$

The proof is left to you as a homework exercise.

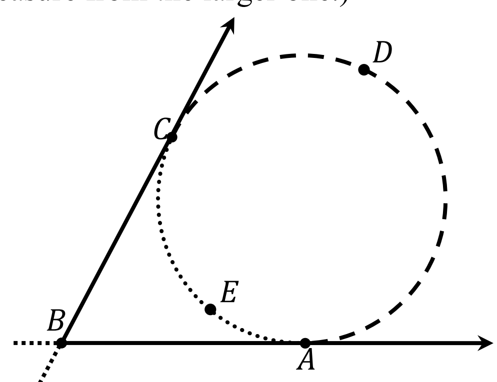


Now on to Type 5 angles. You will justify steps and fill in details of the proof in a homework exercise.

Theorem 145 the angle measure of an angle of Type 5.
The angle measure of an angle of Type 5 can be related to the arc angle measures of the arcs that it intersects in three useful ways:
(i) The angle measure of the angle is equal to 180 minus the arc angle measure of the smaller intercepted arc
(ii) The angle measure of the angle is equal to the arc angle measure of the larger intercepted arc minus 180
(iii) The angle measure of the angle is equal to half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.)

Type 5 angle $\angle ABC$ intercepts dashed arc $\widehat{ADC}$ and also intercepts dotted arc $\widehat{AEC}$. Notice that the angle intercepts two arcs and the arcs share both of their endpoints. The theorem states that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) - \hat{m}(\widehat{AEC})}{2}$$



Proof
(1) Suppose that an angle of Type 5 is given. **(Make a drawing.)**
(2) (Labeling) The two rays of the angle lie in tangent lines that intersect at a point outside the circle. Each ray of the angle intersects the circle. Because the rays lie in tangent lines, we know that each ray intersects the circle exactly once. Label the angle $\angle ABC$ with points A and C being the points of intersection of the angle with the circle. The angle

intercepts a major arc and a minor arc. Let D be a point in the interior of the major arc, and let E be a point in the interior of the minor arc. So the major and minor arcs are denoted by the symbols $\widehat{ADC}$ and $\widehat{AEC}$. **(Update your drawing.)** With this labeling, our goal is to show that the following three equations are true:

(i) $m(\angle ABC) = 180 - \hat{m}(\widehat{AEC})$.

(ii) $m(\angle ABC) = \hat{m}(\widehat{ADC}) - 180$.

(iii) $m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) - \hat{m}(\widehat{AEC})}{2}$.

(3) (Show that equation (i) is true.) Let P be the center of the circle and consider $Polygon(ABCP)$. This is a convex quadrilateral, so its angle sum is 360. **(Justify.)** Observe that $\angle BAP$ and $\angle BCP$ are right angles. **(Justify.) (Update your drawing.)** From here, you should be able to complete the steps to show that equation (i) is true. **(Show the details.)**

(4) (Show that equation (ii) is true.) Determine the relationship between the measure of major arc $\widehat{ADC}$ and minor arc $\widehat{AEC}$. Using this relationship and equation (i), you should be able to show that equation (ii) is true. **(Show the details.)**

(5) (Show that equation (iii) is true.) Add equation (i) and equation (ii) together and divide by 2. **(Show the details.)**

End of proof

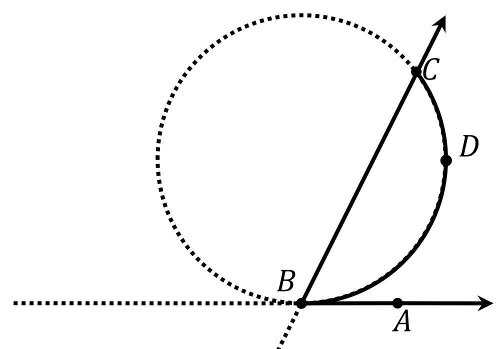
Now on to Type 6 angles. You will justify some of the steps in the proof and fill in some missing steps in a homework exercise.

Theorem 146 the angle measure of an angle of Type 6.

The angle measure of an angle of Type 6 is equal to one half the arc angle measure of the arc intercepted by the angle.

In the picture at right, Type 6 angle $\angle ABC$ intercepts arc $\widehat{BDC}$. Notice that the angle only intercepts one arc. But both endpoints of the arc lie on ray $\overrightarrow{BC}$, and one endpoint of the arc lies one ray $\overrightarrow{BA}$. This is allowed by the definition, because each ray of the angle contains at least one endpoint of the arc. Note that the picture is the special case when angle $\angle ABC$ is acute, but this is not necessarily the case. The theorem states that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC})}{2}$$



Proof

(1) Suppose that an angle of Type 6 is given.

(2) There are three possibilities for the angle.

(i) It is a right angle.

(ii) It is an obtuse angle. That is, its measure is greater than 90.

(iii) It is an acute angle. That is, its measure is less than 90.

Case (i)

(3) Suppose that the angle is a right angle **(Make a drawing.)**

(4) (Labeling) Label the angle $\angle ABC$ where A is a point on the tangent ray and C is the second point of intersection of the secant ray and the circle. Let D be a point in the interior of the arc intercepted by the angle, so that the arc is denoted by the symbols

$\widehat{BDC}$. **(Update your drawing.)** With this labeling, our goal is to show that $m(\angle ABC) = \frac{\widehat{m}(\widehat{BDC})}{2}$.

(5) But $m(\angle ABC) = 90$ because it is a right angle, and $\widehat{m}\left(\widehat{BDC}\right) = 180$ because it is a semicircle. So the equation $m(\angle ABC) = \frac{\widehat{m}(\widehat{BDC})}{2}$ is true. So in this case, the angle measure of the angle of Type 6 is equal to one half the arc angle measure of the arc intercepted by the angle.

Case (ii)

(6) Suppose that the angle is obtuse. **(Make a new drawing.)**

(7) (Labeling) Label the angle $\angle ABC$ where A is a point on the tangent line and C is the second point of intersection of the secant line of the circle.

(8) There exist a point D on the same side of line $\overleftrightarrow{AB}$ as point C such that $m(\angle ABD) = 90$. **(Justify.) (Make a new drawing.)**

(9) Ray $\overrightarrow{AD}$ intersects the circle at a point that we can label E. **(Justify.) (Update your drawing.)** Observe that point E is in the interior of angle $\angle ABC$, and that angle $\angle ABC$ intercepts arc $\widehat{BEC}$. **(Update your drawing.)** With this labeling, our goal is to show that $m(\angle ABC) = \frac{\widehat{m}(\widehat{BEC})}{2}$.

(10) Let F be a point in the interior of the arc intercepted by angle $\angle ABE$, so that the arc is denoted by the symbols $\widehat{BFE}$, and let G be a point in the interior of the arc intercepted by angle $\angle EBC$, so that the arc is denoted by the symbols $\widehat{EGC}$. **(Update your drawing.)**

(11) $m(\angle ABE) = 90 = \frac{\widehat{m}(\widehat{BFE})}{2}$. **(Justify.)**

(12) $m(\angle EBC) = \frac{\widehat{m}(\widehat{EGC})}{2}$. **(Justify.)**

(13) $m(\angle ABC) = \frac{\widehat{m}(\widehat{BFE})}{2} + \frac{\widehat{m}(\widehat{EGC})}{2}$. **(Justify.)**

(14) $m(\angle ABC) = \frac{\widehat{m}(\widehat{BEC})}{2}$. **(Justify.)**

Case (iii)

(15) Suppose that the angle is acute. **(Make a new drawing.)** You should be able to prove that in this case, the angle measure of the angle of Type 6 is equal to one half the arc angle measure of the arc intercepted by the angle. **(Fill in the missing steps, with justifications.)**

Conclusion of cases

(16) We see that in every case, the angle measure of the angle of Type 6 is equal to one half the arc angle measure of the arc intercepted by the angle.

End of proof

Finally, our last proof relating the measures of angles of Types 1 – 7 to the arc angle measure of the arcs that they intersect. We will state and prove a result about Type 7 angles. The proof will make use of the result for Type 6 angles. You will justify the proof steps in a homework exercise.

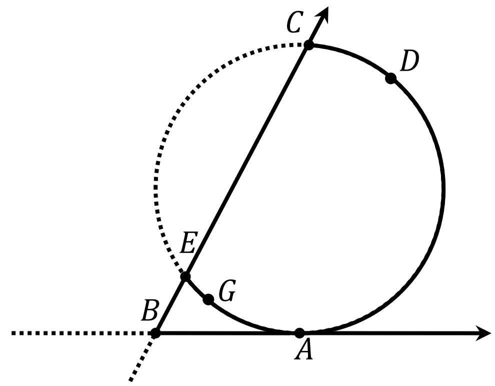
Theorem 147 the angle measure of an angle of Type 7.

The angle measure of an angle of Type 7 is equal to one half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.

Type 7 angle $\angle ABC$ intercepts arc $\widehat{ADC}$ and also intercepts arc $\widehat{AEF}$. Notice that the angle intercepts two arcs and the arcs share an endpoint. The theorem states that

$$m(\angle ABC) = \frac{\hat{m}(\widehat{ADC}) - \hat{m}(\widehat{EGA})}{2}$$

The proof is left to you as a homework exercise.



So far, this chapter has just been about the list of seven types of angles and the arcs that they intersect. We have studied theorems about relationships between their measures. In the next two sections and in the exercises, we will see how these theorems can be put to use to solve interesting geometric problems.

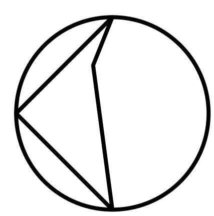
Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 12.6 on page 281.

12.4. Cyclic Quadrilaterals

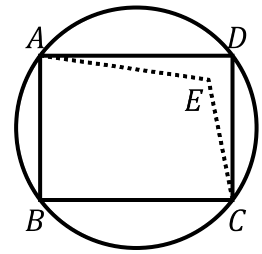
We know from Theorem 107 that in Euclidean Geometry, every triangle can be circumscribed. That is, there exists a circle that passes through all three vertices of the triangle. (Another way of stating this fact is that given any three non-collinear points in Euclidean Geometry, there exists a circle that passes through them.) And the circle is unique: there cannot be two different circles that pass through the same three points. We know this because the center of the circle must be at the point where the perpendicular bisectors of the three sides of the triangle intersect. There is only one such point.

Can the same be said of quadrilaterals in Euclidean Geometry? That is, can every quadrilateral be circumscribed?

It is obvious from pictures that a non-convex quadrilateral cannot be circumscribed. A circle that goes through three of the vertices will not pass through the fourth. We haven't proven this, but we won't bother trying to prove it. Non-convex quadrilaterals are too messy to deal with.



But what about convex quadrilaterals? Can they be circumscribed? Or are there some that can be circumscribed and some that cannot?



A quick answer is, of course there are some that can be circumscribed and some that cannot be circumscribed. In the picture at right, we know that $Quad(ABCE)$ cannot be circumscribed, because the only circle that passes through vertices A, B, C is the one shown, and it does not pass through E.

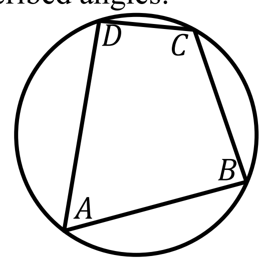
It turns out that there is a simple test to determine whether or not a convex quadrilateral can be circumscribed, and the test is a nice application of the theorems of the previous section.

First, a definition.

Definition 100 cyclic quadrilateral

A quadrilateral is said to be cyclic if the quadrilateral can be circumscribed. That is, if there exists a circle that passes through all four vertices of the quadrilateral.

We start by considering a quadrilateral that is known to be cyclic. Here is a theorem that you will prove in the exercises. Its proof uses Theorem 141 about the measure of inscribed angles.



Theorem 148 In Euclidean Geometry, in any cyclic quadrilateral, the sum of the measures of each pair of opposite angles is 180. That is, if $Quad(ABCD)$ is cyclic, then $m(\angle A) + m(\angle C) = 180$ and $m(\angle B) + m(\angle D) = 180$.

An obvious question is, does the theorem just presented work in reverse? That is, if it is known that a quadrilateral has the property that the sum of the measures of each pair of opposite angles is 180, then will the quadrilateral definitely be cyclic? The answer will turn out to be yes, but we need to prove that answer as a theorem. To do that, we will need a preliminary theorem.

Theorem 149 about angles that intercept a given arc

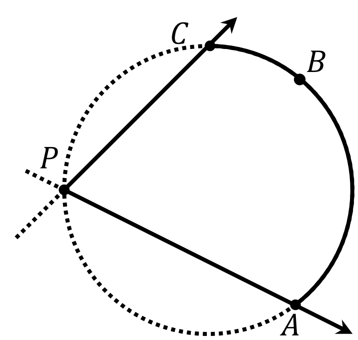
In Euclidean geometry, given an arc $\widehat{ABC}$ and a point P on the opposite side of line $\overleftrightarrow{AC}$ from point B, the following are equivalent:

(1) Point P lies on $Circle(A, B, C)$.

(2) $m(\angle APC) = \frac{\hat{m}(\widehat{ABC})}{2}$.

Proof

In Euclidean geometry, given an arc $\widehat{ABC}$ and a point P on the opposite side of line $\overleftrightarrow{AC}$ from point B, the angle $\angle APC$ will intercept arc $\widehat{ABC}$.



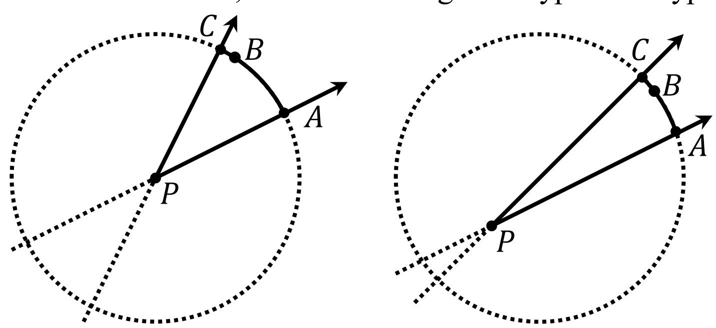
Show that (1) ➔ (2).

Suppose that (1) is true. That is, suppose that point P lies on $Circle(A, B, C)$. Then $\angle APC$ will be an inscribed angle. Theorem 141 about the measure of inscribed angles tells us that $m(\angle APC) = \frac{\hat{m}(\widehat{ABC})}{2}$, So (2) is true.

Show that ~(1) ➔ ~(2).

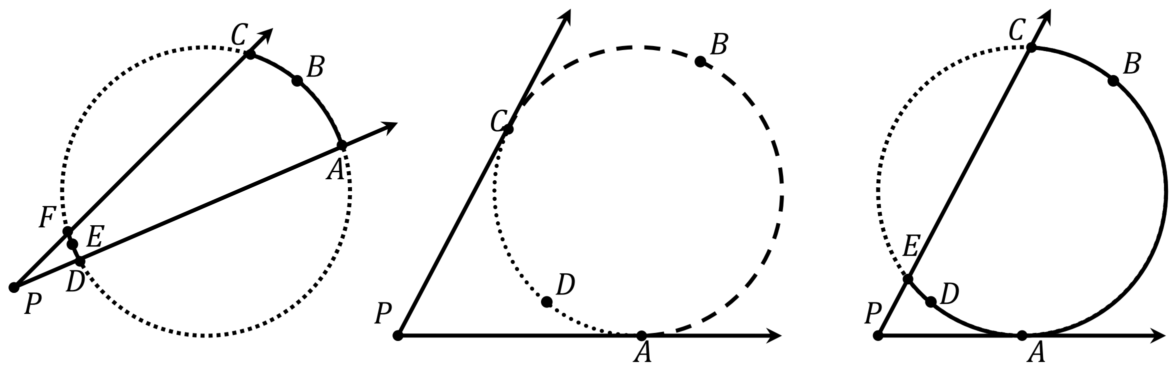
Suppose that (1) is false. That is, suppose that point P does not lie on $Circle(A,B,C)$. Then P lies in the interior of the circle or in the exterior.

If P lies in the interior of the circle, then it is an angle of Type 1 or Type 3.



In either case, $m(\angle APC) > \frac{\widehat{m}(\widehat{ABC})}{2}$. **(Justify)** So statement (2) is false.

If P lies in the exterior of the circle, then it is an angle of Type 4, 5, or 7.



In either case, $m(\angle APC) < \frac{\widehat{m}(\widehat{ABC})}{2}$. **(Justify)** So statement (2) is false.

We see that in all cases, statement (2) is false.

End of proof

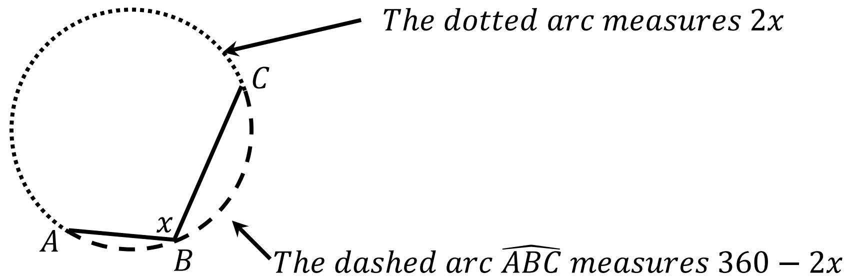
With Theorem 149 at our disposal, it is now possible to prove a theorem that answers the question that we posed earlier: If it is known that a quadrilateral has the property that the sum of the measures of each pair of opposite angles is 180, then will the quadrilateral definitely be cyclic? We discussed the fact that the answer would turn out to be yes. Here is the theorem.

Theorem 150 In Euclidean Geometry, in any convex quadrilateral, if the sum of the measures of either pair of opposite angles is 180, then the quadrilateral is cyclic.

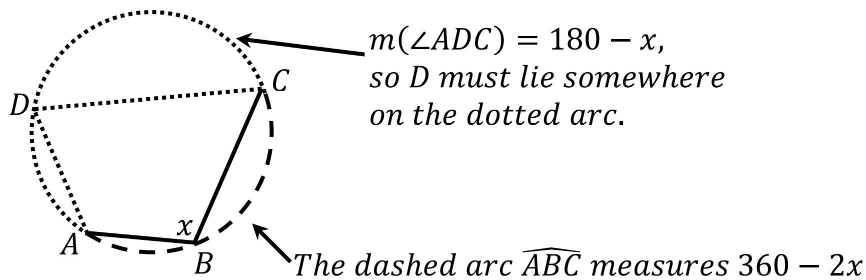
Proof

In Euclidean geometry, suppose that a convex quadrilateral has the property that the sum of the measures of one of its pairs of opposite angles is 180. Label the vertices A, B, C, D so that it is angles $\angle B$ and $\angle D$ whose measures add up to 180. Let $x = m(\angle B) = m(\angle ABC)$, and let $y = m(\angle D) = m(\angle CDA)$.

For now, forget about point D and consider just points A, B, C. By Theorem 107, there exists a $Circle(A,B,C)$ that passes through these three points. The picture below illustrates such a circle. In the picture, angle $\angle ABC$ intercepts a dotted arc. By Theorem 141 about the measure of inscribed angles, the arc angle measure of the dotted arc must be $2x$. By Theorem 138, the arc angle measure of the dashed arc $\widehat{ABC}$ that makes up the rest of the circle must be $360 - 2x$.



Now consider the dashed arc $\widehat{ABC}$. Observe that $\frac{\hat{m}(\widehat{ABC})}{2} = \frac{360-2x}{2} = 180 - x$. This is the number that is the known measure of $\angle D$ in our quadrilateral. Since point D lies on the opposite side of line $\overleftrightarrow{AC}$ from point B and has the property that $m(\angle ADC) = 180 - x = \frac{\hat{m}(\widehat{ABC})}{2}$, Theorem 149 tells us that point must D lie somewhere on the dotted arc.



So point D must lie on $Circle(A, B, C)$. In other words, $Quadrilateral(ABCD)$ is cyclic!

End of proof

Before going on to read the next section, you should do the exercises for the current section. The exercises are found in Section 12.6 on page 281.

12.5. The Intersecting Secants Theorem

So far in this chapter, our theorems have been about the measure of angles that intersect circles. In this final section of the chapter, we will turn our attention to the lengths of chords and other segments on secant lines. We will will only study three theorems. The key to all three is this first theorem. You will see that its proof uses similarity.

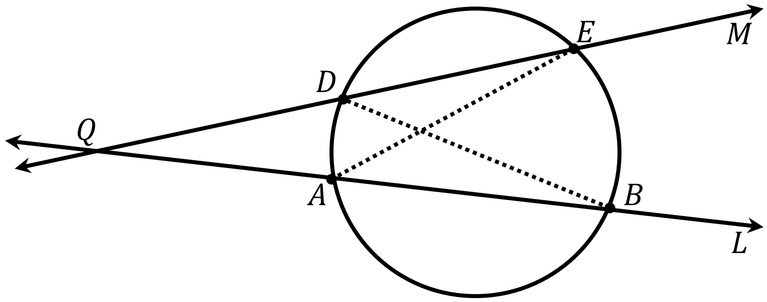
Theorem 151 The Intersecting Secants Theorem

In Euclidean geometry, if two secant lines intersect, then the product of the distances from the intersection point to the two points where one secant line intersects the circle equals the product of the distances from the intersection point to the two points where the other secant line intersects the circle. That is, if secant line L passes through a point Q and intersects the circle at points A and B and secant line M passes through a point Q and intersects the circle at points D and E, then $QA \cdot QB = QD \cdot QE$.

Proof

Case 1: Q is outside the circle.

(1) Suppose that Q is outside the circle. Then label the two intersection points of the circle and line L as A, B so that $Q * A * B$. And label the two intersection points of the circle and line M as D, E so that $Q * D * E$.



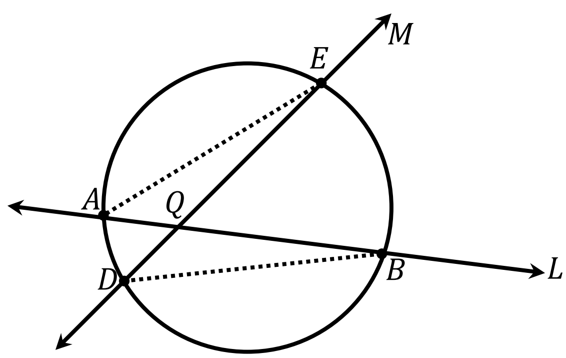
(2) Observe that $\angle QBD \cong \angle QEA$. **(Justify.)**
(3) Therefore, $\Delta QBD \sim \Delta QEA$. **(Justify.) (Make a new drawing of the separate triangles.)**
(4) So $\frac{QB}{QE} = \frac{QD}{QA}$.
(5) Therefore, $QA \cdot QB = QD \cdot QE$.

Case 2: Q is on the circle.
(6) Suppose that Q is on the circle. Then label the two intersection points of the circle and line L as A, B so that $Q = A$. And label the two intersection points of the circle and line M as D, E so that $Q = D$.
(7) Observe that in this case, $QA = 0 = QD$, so the equation

$$QA \cdot QB = QD \cdot QE$$

becomes the trivial equation $0 = 0$, which is true.

Case 3: Q is inside the circle but not at the center.
(8) Suppose that Q is inside the circle but not at the center. Then label the two intersection points of the circle and line L as A, B so that $A * Q * B$. And label the two intersection points of the circle and line M as D, E, it doesn't matter the order.



(9) Fill in the missing details to show that the equation

$$QA \cdot QB = QD \cdot QE$$

is true in this case as well.

Case 4: Q is at the center of the circle.
(10) In this case, the four segments $\overline{QA}, \overline{QB}, \overline{QD}, \overline{QE}$ are all radial segments, with length r, so the equation

$$QA \cdot QB = QD \cdot QE$$

becomes the trivial equation $r^2 = r^2$, which is true.

End of proof

The final theorem of the chapter has a statement that is believable. You will be asked to provide a proof in the exercises.

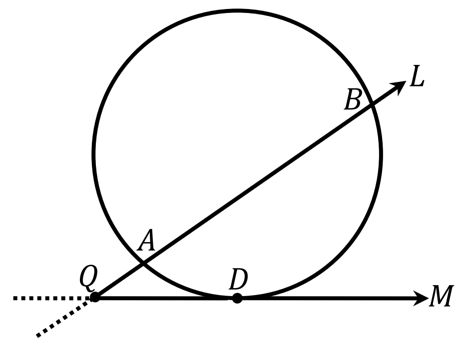
Theorem 152 about intersecting secant and tangent lines.

In Euclidean geometry, if a secant line and tangent line intersect, then the square of the distance from the point of intersection to the point of tangency equals the product of the distances from the intersection point to the two points where the secant line intersects the circle. That is, if a secant line passes through a point Q and intersects the circle at points A and B and a tangent line passes through Q and intersects the circle at D, then

$$(QD)^2 = QA \cdot QB$$

In the figure at right, secant line L intercepts the circle at points A and B. Line M is tangent to the circle at point D. Lines L and M intersect at point Q. The theorem states that $(QD)^2 = QA \cdot QB$.

The proof is left to you as a homework exercise.



12.6. Exercises for Chapter 12

Exercises for Section 12.1 Circular Arcs (Section starts on page 263.)

[1] The notion of an angle intercepting an arc was made precise in Definition 97 (angle intercepting an arc, found on page 265). In the discussion of the Type 7 angle, it is written that angle $\angle ABC$ intercepts arc $\widehat{ADC}$ and also intercepts arc $\widehat{AEF}$. Why does angle $\angle ABC$ *not* intercept arc $\widehat{FAC}$? Explain.

Exercises for Section 12.2 Angle Measure of an Arc (Section starts on page 267.)

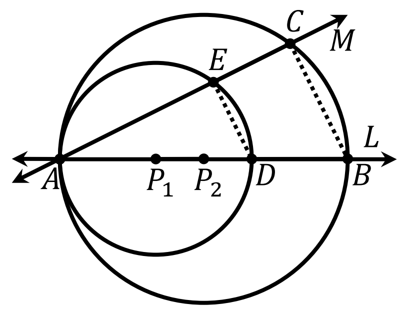
[2] Prove Theorem 139: Two chords of a circle are congruent if and only if their corresponding arcs have the same measure.

[3] (Advanced) Provide drawings and fill in the missing part for the proof of Theorem 140 (The Arc Measure Addition Theorem) (found on page 268).

Exercises for Section 12.3 The Measure of Angles Related to the Measures of Arcs that they Intercept (Section starts on page 270.)

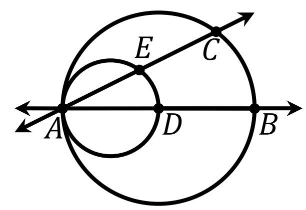
[4] Justify the steps and provide drawings for the proof of Theorem 141 (the angle measure of an inscribed angle (Type 2) is equal to half the arc angle measure of the intercepted arc.) (found on page 270).

[5] Tangent circles were defined in Definition 69. They are circles that intersect in exactly one point. In Section 8.6 Exercises [19] and [20], it was proven that two tangent circles share a tangent line at their point of tangency, and that the centers of the two circles and their point of tangency are collinear. In this problem, consider two circles called $Circle(P_1, r)$ and $Circle(P_2, R)$ that are tangent at point A. $Circle(P_1, r)$ is inside $Circle(P_2, R)$. That is, center P_1 is in the interior of $Circle(P_2, R)$, and $r < R$. By the result cited above, points A, P_1, P_2 are collinear on a line that we can call L. This line is a secant line for both circles. Its second point of intersection with $Circle(P_1, r)$ is labeled D; its second point of intersection with $Circle(P_2, R)$ is labeled B. A second secant line M intersects $Circle(P_1, r)$ at points A and E and intersects $Circle(P_2, R)$ at points A and C. Prove that $\Delta ABC \sim \Delta ADE$.



[6] In the drawing for the previous problem, suppose that $AE = 8$ and $EC = 4$ and $DE = 6$. Find BC and the two radii r and R.

[7] In the drawing at right, the two circles are tangent. D is the midpoint of segment $\overline{AB}$. Prove that E is the midpoint of segment $\overline{AC}$.



[8] Justify the steps and provide drawings for the proof of Theorem 143 (the angle measure of an angle of Type 3.) (found on page 272).

[9] Prove Theorem 144 (the angle measure of an angle of Type 4.) (found on page 273).

[10] Justify the steps, fill in missing details, and provide drawings for the proof of Theorem 145 (the angle measure of an angle of Type 5.) (found on page 273).

[11] Justify the steps, fill in missing details, and provide drawings for the proof of Theorem 146 (the angle measure of an angle of Type 6.) (found on page 274).

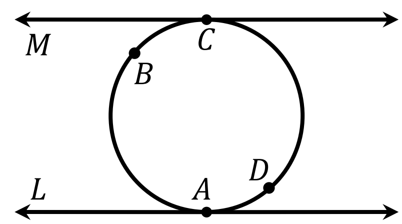
[12] Parallel lines L and M both intersect a circle. There are two arcs that have the following properties:

- One endpoint of the arc lies on each line.
- The interior of the arc lies between the lines.

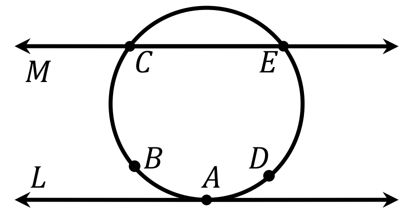
We will say that these two arcs are intercepted by the parallel lines L and M. Prove that the two arcs have the same measure.

There are three possible cases. Use the following notation.

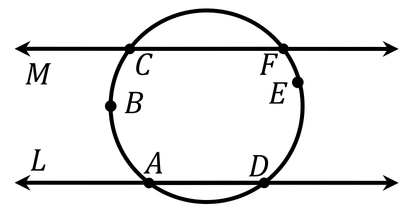
Case 1: Lines L and M are both tangent. The two intercepted arcs can be labeled $\widehat{ABC}$ and $\widehat{ADC}$.
Prove that $\hat{m}(\widehat{ABC}) = \hat{m}(\widehat{ADC})$.
Hint: Draw segment $\overline{AC}$.



Case 2: Lines L is a tangent line and line M is a secant line. The two intercepted arcs can be labeled $\widehat{ABC}$ and $\widehat{ADE}$.
Prove that $\hat{m}(\widehat{ABC}) = \hat{m}(\widehat{ADE})$.
Hint: Draw segment $\overline{AC}$.



Case 3: Both lines are secant lines. The two intercepted arcs can be labeled $\widehat{ABC}$ and $\widehat{DEF}$.
Prove that $\hat{m}(\widehat{ABC}) = \hat{m}(\widehat{DEF})$.
Hint: Draw segment $\overline{DC}$.



[13] Prove Theorem 147 (the angle measure of an angle of Type 7.) (found on page 275).

Exercises for Section 12.4 Cyclic Quadrilaterals (Section starts on page 276.)

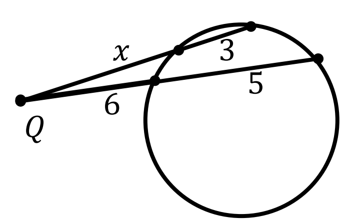
[14] Prove Theorem 148 (In Euclidean Geometry, in any cyclic quadrilateral, the sum of the measures of each pair of opposite angles is 180. That is, if $Quad(ABCD)$ is cyclic, then $m(\angle A) + m(\angle C) = 180$ and $m(\angle B) + m(\angle D) = 180$.) (found on page 277).

Exercises for Section 12.5 The Intersecting Secants Theorem (Section starts on page 279.)

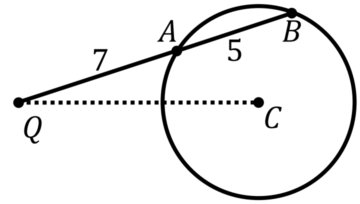
[15] Justify the steps and fill in the missing parts of the proof of Theorem 151 (The Intersecting Secants Theorem) (found on page 279).

[16] Prove Theorem 152 (about intersecting secant and tangent lines.) (found on page 281). Make lots of drawings.
Hint: Study the proof of Theorem 151 (The Intersecting Secants Theorem) (found on page 279). Notice the key to the proof was to identify two similar triangles. They were known to be similar by the AA Similarity Theorem. One of the pairs of congruent angles was known to be congruent because they were either the same angle or vertical angles. The other pair of congruent angles was known to be congruent because they were both inscribed angles and they intercepted the same arc. The same sort of strategy will work for this theorem.

[17] The figure at right is not drawn to scale. Find x.



[18] In the figure at right, the circle has radius $r = 4$. If $QA = 7$ and $AB = 5$, find QC.

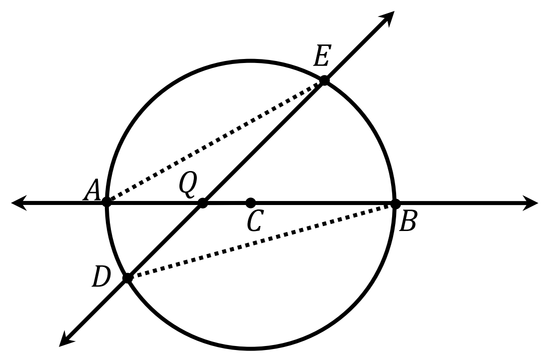


[19] In the proof of Theorem 151 Cases 1 and 3, similar triangles are used. In both cases, the proof uses the equality of the ratios of the lengths of two pairs of corresponding sides that all have the point Q as one of their endpoints. It is reasonable to wonder if anything useful can be made of the fact that the ratio of the lengths of the third pair of corresponding sides (the sides that don't touch point Q) is also the same. That is, in those figures,

$$\frac{QB}{QE} = \frac{QD}{QA} = \frac{BD}{EA}$$

Here is a rockin problem that uses the ratio of the lengths of the third sides. I did not make this problem up. It was created by a mathematician named Vasil'ev and appeared in the Russian magazine *Kvant* (M26, March-April, 1991, 30). (According to Wikipedia, *Kvant* (Russian: Квант for "quantum") is a popular science magazine in physics and mathematics for school students and teachers, issued since 1970 in Soviet Union and continued in Russia.) It was also included in the book *Mathematical Chestnuts from Around the World*, by R. Honsberger.

Boat 1 and boat 2, which travel at constant speeds, not necessarily the same speed, depart at the same time from docks A and D, respectively, on the banks of a circular lake. If they go straight to docks B and E, respectively, they collide. Prove that if boat 1 goes instead straight to dock E and boat 2 goes straight to dock B, they arrive at their destinations simultaneously.



Hints: (Okay, this is really a complete outline of the solution.)
(1) Let v_1 and v_2 be the speeds of boat 1 and boat 2.
(2) The time it takes boat 1 to get from A to Q is $\frac{AQ}{v_1}$. The time it takes boat 2 to get from D to Q is $\frac{DQ}{v_2}$. These two times must be equal, because the boats will collide at Q.
(3) The time it takes boat 1 to get from A to E is $\frac{AE}{v_1}$. The time it takes boat 2 to get from D to B is $\frac{DB}{v_2}$. The goal is to prove that these two times are equal.
(4) Because the times in steps must be equal, the two ratios must be equal. Set the two ratios equal and solve for the ratio $\frac{v_1}{v_2}$. The result should be an equation involves the ratios $\frac{v_1}{v_2}$ and $\frac{AQ}{DQ}$.
(5) Using facts about angles and arcs, identify similar triangles.
(6) Find a realtionship between the the ratios $\frac{AQ}{DQ}$ and $\frac{AE}{DB}$.
(7) From results of (4) and (6), find an equation giving a relationship between ratios $\frac{v_1}{v_2}$ and $\frac{AE}{DB}$.
(8) Rearrange the equation from (7) to get a new equation expressing a relationship between the ratios $\frac{AE}{v_1}$ and $\frac{DB}{v_2}$.

13. Euclidean Geometry VI: Advanced Triangle Theorems

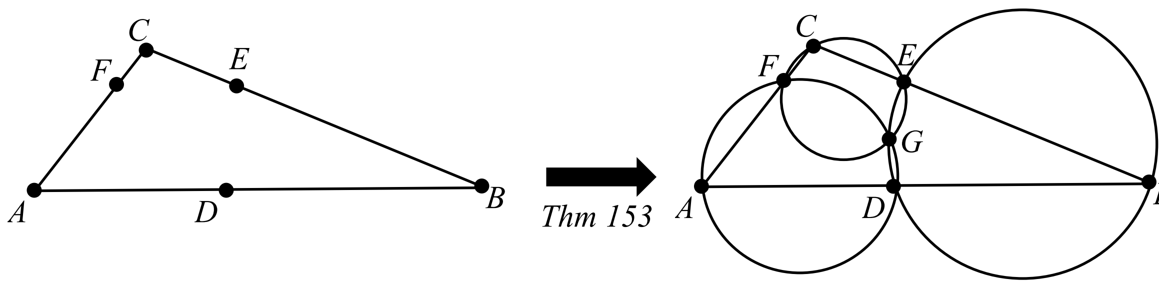
This short chapter presents some advanced theorems about triangles in Euclidean Geometry. You will be asked to prove them.

13.1. Six Theorems

Our first theorem is about the existence of a point of intersection of three circles attached to a triangle.

Theorem 153 Miquel's Theorem

If points A, B, C, D, E, F are given such that points A, B, C are non-collinear and $A * D * B$ and $B * E * C$ and $C * F * A$, then $Circle(A, D, F)$ and $Circle(B, E, D)$ and $Circle(C, F, E)$ exist and there exists a point G that lies on all three circles.



The next theorem is about a surprising relationship between the collinearity of three points on the lines determined by the sides of a triangle and the ratios of the lengths of certain segments defined by those points.

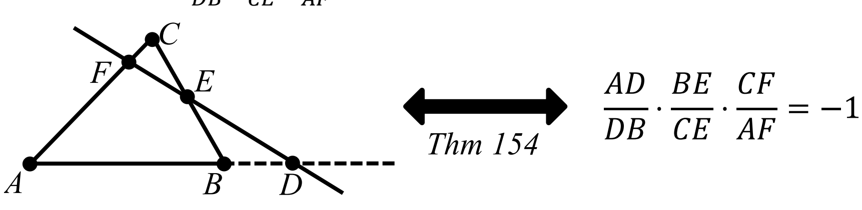
Theorem 154 Menelaus's Theorem

Given: points A, B, C, D, E, F such that A, B, C are non-collinear and $A * B * D$ and $B * E * C$ and $C * F * A$

Claim: The following are equivalent

(1) Points D, E, F are collinear.

(2) $\frac{AD}{DB} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = -1$



The next theorem is about a surprising relationship between the concurrence of three lines determined by points on the sides of a triangle and the ratios of the lengths of certain segments determined by those points.

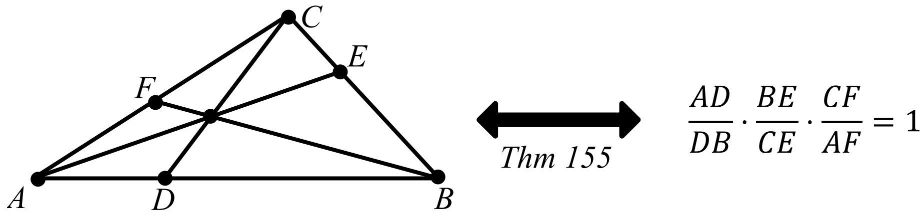
Theorem 155 Ceva's Theorem

Given: points A, B, C, D, E, F such that A, B, C are non-collinear and $A * D * B$ and $B * E * C$ and $C * F * A$

Claim: The following are equivalent

(1) Lines $\overleftrightarrow{AE}, \overleftrightarrow{BF}, \overleftrightarrow{CD}$ are concurrent

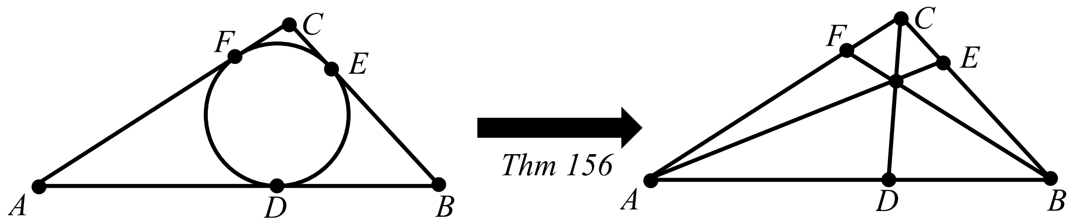
(2) $\frac{AD}{DB} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = 1$



The next theorem is a corollary to Ceva's Theorem.

Theorem 156 (Corollary to Ceva's Theorem)

If A, B, C are non-collinear points and $A * D * B$ and $B * E * C$ and $C * F * A$ are the points of tangency of the inscribed circle for ΔABC, then lines $\overleftrightarrow{AE}, \overleftrightarrow{BF}, \overleftrightarrow{CD}$ are concurrent.



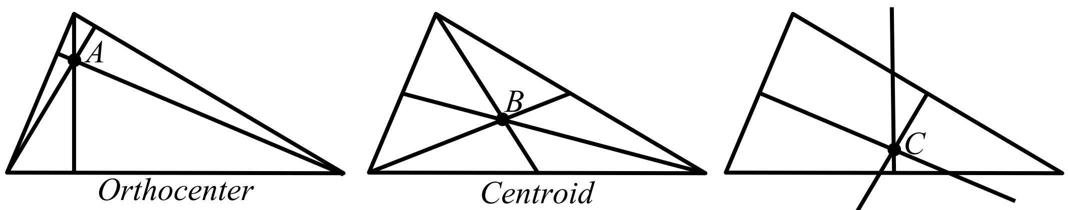
Recall our three important theorems about concurrence of each of three kinds of important lines associated to triangles.

- In Theorem 106 on page 214, we proved that the perpindicular bisectors of the three sides of any triangle are concurrrent. In Definition 71 we gave the name *circumcenter* to that point of concurrence.
- In Theorem 113 on page 219, we proved that the three altitudes of any triangle are concurrent. Definition 76 gave the name *orthocenter* to that point of concurrence.
- In Theorem 116 on page 220, we proved that the three medians of any triangle are concurrent. Definition 78 gave the name *centroid* to that point of concurrence.

Of course, if a triangle is equilateral, then the three kinds of important lines are all the same lines (Can you prove this? The proof would be similar to the proof of Theorem 85 on page 199), so the circumcenter, orthocenter, and centroid will all be the same point.

The following surprising theorem is about the case where the triangle is *not* equilateral.

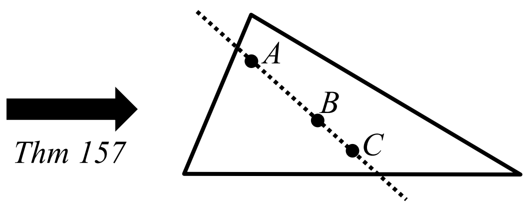
Theorem 157 Collinearity of the orthcenter, centroid, and circumcenter
If a non-equilateral triangle's orthcenter, centroid, and circumcenter are labeled A, B, C,
Then A, B, C are collinear, with $A * B * C$ and $AB = 2BC$.



Orthocenter (point of concurrence of altitudes)

Centroid (point of concurrence of medians)

Circumcenter (point of concurrence of perpendicular bisectors)



Definition 101 The Euler Line of a non-equilateral triangle
Given a non-equilateral triangle, the Euler Line is defined to be the line containing the orthocenter, centroid, and circumcenter of that triangle. (Existence and uniqueness of this line is guaranteed by Theorem 157.)

The statement of our final theorem of the section is made simpler if we first introduce a new kind of special point.

Definition 102 The three Euler Points of a triangle are defined to be the midpoints of the segments connecting the vertices to the orthocenter.

Given any triangle, consider the following three sets of three special points and the three special circles that they define.

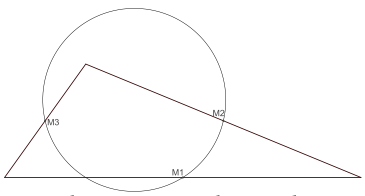
- The midpoints of the three sides. These could be labeled M_1, M_2, M_3. These three points are non-collinear (Can you explain why?), so there exists a circle that passes through all three. It would be denonted $Circle(M_1, M_2, M_3)$
- The feet of the three altitudes. These could be labeled F_1, F_2, F_3. These are the points where the altitude lines intersect the lines defined by the vertices. The three feet are non-collinear (Can you explain why?), so as with the three midoints, there exists a circle that passes through all three feet. It would be denoted $Circle(F_1, F_2, F_3)$.
- The feet of the three Euler points. These could be labeled E_1, E_2, E_3. The three Euler points are non-collinear (Can you explain why?), so as with the three midoints and the three feet, there exists a circle that passes through all three Euler points. It would be denoted $Circle(E_1, E_2, E_3)$.

It is an astonishing fact that the three special circles described above are in fact the same circle. That is, there exists a single circle that passes through all nine special points. This fact is articulated in the following theorem.

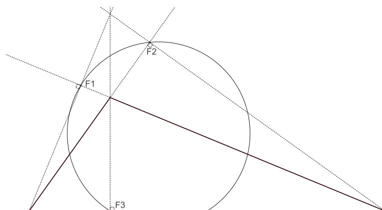
Theorem 158 Existence of a circle passing through nine special points associated to a triangle
For every triangle, there exists a single circle that passes through the midpoints of the three sides, the feet of the three altitudes, and the three Euler points.

Definition 103 The *nine point circle* associated to a triangle is the circle that passes through the midpoints of the three sides, the feet of the three altitudes, and the three Euler points. (The existence of the nine point circle is guaranteed by Theorem 158.)

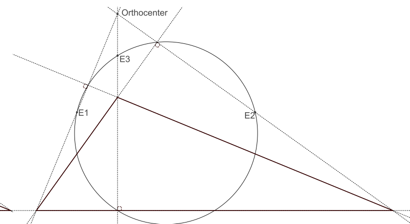
The drawing below illustrates the statement of Theorem 158.



circle passing through the midpoints of the three sides

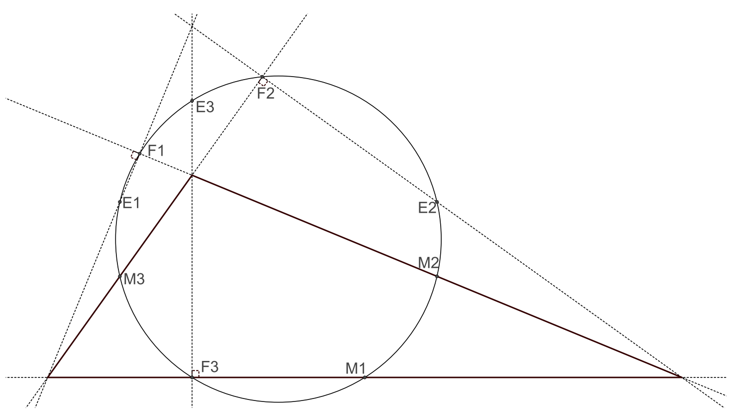


circle passing through the feet of the three altitudes



circle passing through the three Euler points

Thm 158



The three circles are actually the same circle, called the nine point circle,

13.2. Exercises for Chapter 13

The six theorems in this chapter are all advanced. (One clue that they are advanced is that five of them are named for the mathematicians to whom the first proof is attributed. That only happens for advanced theorems.) But all seven can be proven using the definitions and theorems that you have studied in previous chapters. Write proofs of the seven theorems. You are encouraged to collaborate with others, and to look to the internet for clues and sample proofs. But ultimately, your proof should be written using references to defined terms and theorems from this book..

14. The Circumference and Area of Circles

At some point in junior high and high school, you learned the formulas for the circumference C and area A of a circle of radius r and diameter $d = 2r$:

$$C = \pi d = 2\pi r$$
$$A = \pi r^2$$

You learned these as given rules, rules that could be applied to solve geometric problems. We would like to fit those formulas into what we have learned so far. But there are two obvious questions:

(1) What is π?
(2) How do we know that the formulas above are true?

But there is a third, less obvious question:

(3) What do we mean by the circumference of a circle, or the area of a circular region?

In this chapter, we will discuss these questions. We will answer them in reverse order.

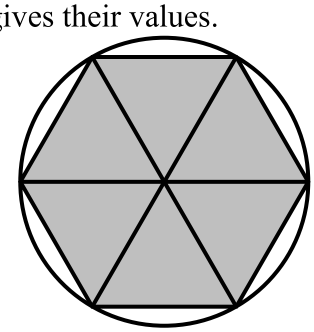
14.1. Defining the Circumference and Area of a Circle

We have a notion of length in our geometry: The length of a line segment is defined to be the distance between the endpoints. We generalize this for the length of a segmented path: the length of the segmented path is the sum of the lengths of the line segments that make up that path. But we have never discussed a notion of length for curvy objects. Notice that when we measured arcs in the previous chapter, it was arc angle measure that we were measuring, not arc length. In this section, we will introduce the circumference of a circle. It will be a number obtained as the limit of a sequence of numbers that are lengths of segmented paths.

For now, consider a circle with radius $r = \frac{1}{2}$ and with a regular polygon inscribed inside. We will be interested in regular n-gons with $n = 6, 12, 24, 48, 96, ...$ In other words, $n = 3 \cdot 2^k$ for $k = 1, 2, 3, 4, ...$

The first case, $k = 1$, is shown in the diagram at right below. The table at left below introduces notation for some quantities that will be useful to us, and gives their values.

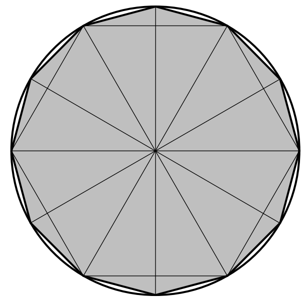
Value of $\boldsymbol{k}$: $k = 1$
Name of polygonal region: $Poly_1$
Number of sides: $n = 3 \cdot 2^k = 3 \cdot 2^1 = 6$
Length of sides: $x_1 = \frac{1}{2}$.
Perimeter: $P_1 = 3$
Area: $A_1 = \frac{3\sqrt{3}}{8} \approx 0.65$



The value of the perimeter is easy: Six sides of length $x_1 = \frac{1}{2}$ each. For the area, we used the fact that an equilateral triangle that has sides of length x will have a height of $h = \frac{x\sqrt{3}}{2}$.

To get a polygon for the second case, $k = 2$, we start with the figure above, then draw the three lines that pass through the center of the circle and through the midpoints of the sides of the hexagon. We put points at the intersection of these lines and the circle. These six new points, along with the six vertices of the original hexagon, will be the twelve vertices of the new polygon.

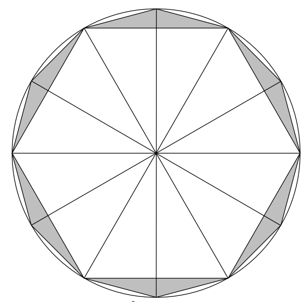
Value of k: $k = 2$
Name of polygonal region: $Poly_2$
Number of sides: $n = 3 \cdot 2^k = 3 \cdot 2^2 = 12$
Length of sides: $x_2 =?.$
Perimeter: $P_2 =?$
Area: $A_2 =?$



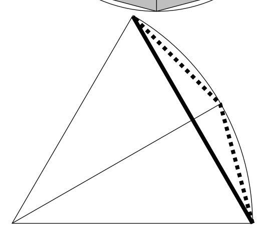
It would not be terribly difficult to compute the length of the sides for this region, or to compute its area. But those values are not important.

What is important is to notice that the area A_2 will be greater than the area A_1. We see that from the figure at right. The shaded region is part of $Poly_2$ but not part of $Poly_1$. Therefore,

$$\begin{aligned} Area(Poly_2) &= Area(Poly_1 \cup shaded\ region) \\ &= Area(Poly_1) + Area(shaded\ region) \\ &> Area(Poly_1) \end{aligned}$$



It is also important to notice that the perimeter P_2 will be greater than the perimeter P_1. We see that from the figure at right. The solid dark line is part of the perimeter of $Poly_1$, while the dotted dark line is part of the perimeter of $Poly_2$.



Even more important is to realize that the method that we used to construct $Poly_2$ from $Poly_1$ will be the same method that we will use in general to construct $Poly_{k+1}$ from $Poly_k$. To get $Poly_{k+1}$, we start with $Poly_k$ and draw the lines that pass through the center of the circle and through the midpoints of the sides of $Poly_k$. We put points at the intersection of these lines and the circle. These new points, along with the vertices of $Poly_k$, will be the vertices of the new $Poly_{k+1}$. As a result, there will be pictures analogous to the two above that will show that $P_{k+1} > P_k$ and $A_{k+1} > A_k$.

So we have a list of regular polygons inscribed in the circle. Their polygonal regions are called

$$Poly_1, Poly_2, Poly_3, \dots, Poly_k, Poly_{k+1}, \dots$$

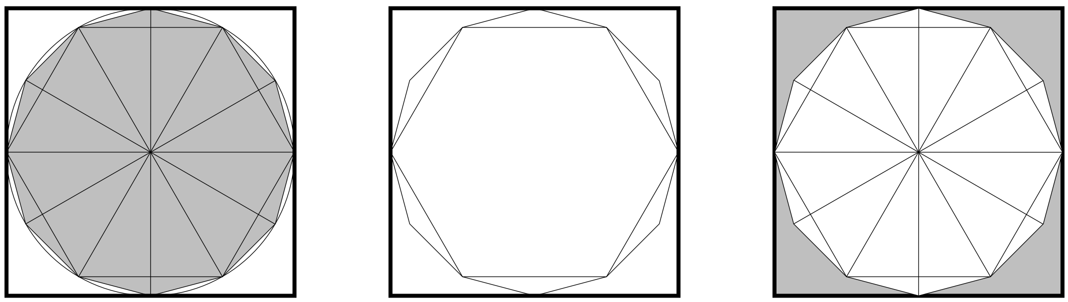
Their perimeters form a strictly increasing sequence of real numbers

$$3 = P_1 < P_2 < P_3 < \cdots < P_k < P_{k+1} < \cdots$$

and their areas form a strictly increasing sequence of real numbers

$$\frac{3\sqrt{3}}{8} = A_1 < A_2 < A_3 < \cdots < A_k < A_{k+1} < \cdots$$

Also, observe that the sequence of perimeters is bounded above by the number 4 and the sequence of areas is bounded above by the number 1. To see why, consider a square tangent to the circle as shown in the figure at left below. Once we have the square, it helps to omit the circle and some lines as shown in the figures in the middle and at right below.



The square has sides of length 1 because that is the length of a diameter of the circle. So the square has perimeter 4 and area 1. Looking at the middle drawing, it is very believeable that each polygonal region $Poly_k$ has perimeter $P_k < 4$. But to be thorough about it, one should demonstrate that each polygonal path is a shortcut compared to the path around the square. We won't do this. Looking at the drawing on the right, we see that the square region can be considered as a union of the polygonal region $Poly_k$ and the shaded region outside $Poly_k$. But the shaded region is a polygonal region with some positive area. So the area of the square is strictly greater than the area of $Poly_k$. Therefore, each polygonal region $Poly_k$ has area $A_k < 1$.

Recall that any sequence of real numbers that is increasing and bounded above will have a limit. So, there exists a real number that is the $\lim_{k\to\infty} P_k$ and a there also exists a real number that is the $\lim_{k\to\infty} A_k$. It is these real numbers that we will define to be the circumference of the circle and the area of the circular region. Before doing that, note that all of our analysis has been for a circle of diameter 1. The same sort of analysis could be done for a general circle of diameter d.

Definition 104 circumference of a circle and area of a circular region

Given a circle of diameter d, for each $k = 1,2,3, \ldots$ define $Poly_k$ to be a polygonal region bounded by a regular polygon with $3 \cdot 2^k$ sides, inscribed in the circle. Define P_k and A_k to be the perimeter and area of the k^{th} polygonal region. The resulting sequences $\{P_k\}$ and $\{A_k\}$ are increasing and bounded above and so they each have a limit.

- Define the circumference of the circle to be the real number $C = \lim_{k\to\infty} P_k$.
- Define the area of the circular region to be the real number $A = \lim_{k\to\infty} A_k$.

With that definition, we have answered the third question that was posed on page 289:

(3) What do we mean by the circumference of a circle, or the area of a circular region?

But that definition does not tell us much about the values of the real numbers that are the circumference and area. We will consider the values of those numbers in the next section.

14.2. Estimating the Value of the Circumference and Area of a Circle

In the previous section, we defined the real numbers that are the circumference of a circle and the area of a circular region, but our definition did not mention the values of these numbers. How big are they?

We should note that we can state some crude estimates based on work that we have already done.

Let's start by estimating the circumference. For the circumference of a circle of diameter $d = 1$, we saw that the perimeters of the inscribed polygons form a strictly increasing sequence of real numbers

$$3 = P_1 < P_2 < P_3 < \cdots < P_k < P_{k+1} < \cdots$$

and that this sequence was bounded above by the number 4. Therefore, the circumference $C = \lim_{k\to\infty} P_k$ will be bounded by $3 < C \le 4$.

More generally, for the circumference of a circle of diameter d, the perimeters of the inscribed polygons will form a strictly increasing sequence of real numbers

$$3d = P_1 < P_2 < P_3 < \cdots < P_k < P_{k+1} < \cdots$$

and this sequence will be bounded above by the number $4d$. Therefore, the circumference $C = \lim_{k\to\infty} P_k$ will be bounded by $3d < C \le 4d$.

Now let's estimate the area. For the area of a circle of diameter $d = 1$, we saw that the areas of the inscribed polygons form a strictly increasing sequence of real numbers

$$\frac{3\sqrt{3}}{8} = A_1 < A_2 < A_3 < \cdots < A_k < A_{k+1} < \cdots$$

and that this sequence was bounded above by the number 1. Therefore, the area $A = \lim_{k\to\infty} A_k$ will be bounded by $\frac{3\sqrt{3}}{8} < A \le 1$.

More generally, for the area of a circle of diameter d, the areas of the inscribed polygons will form a strictly increasing sequence of real numbers

$$\frac{3\sqrt{3}}{8}d^2 = A_1 < A_2 < A_3 < \cdots < A_k < A_{k+1} < \cdots$$

and this sequence will be bounded above by the number d^2. Therefore, the area $A = \lim_{k\to\infty} A_k$ will be bounded by $\frac{3\sqrt{3}}{8}d^2 < A \le d^2$. In terms of the radius, the bounds are $\frac{3\sqrt{3}}{2}r^2 < A \le 4r^2$. Note that $\frac{3\sqrt{3}}{2} \approx 2.598$, so we have the bounds $2.5r^2 < A \le 4r^2$

14.3. Introducing Pi

We would like to have a more precise estimate of the circumference and area, more precise than the crude estimates that we have found so far.

We will start by trying to make precise the relationship between the circumference and the diameter and also make precise the relationship between the area and the diameter (or radius).

The following theorem will get us started.

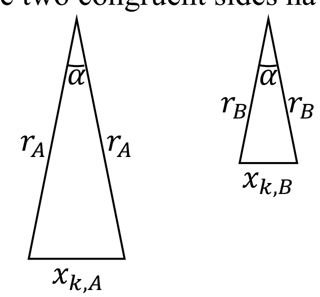
Theorem 159 The ratio $\frac{circumference}{diameter}$ is the same for all circles.

Proof:
Consider two different circles A and B of radius r_A and r_B and diameter $d_A = 2r_A$ and $d_B = 2r_B$ and circumference C_A and C_B. With this notation, our goal is to show that

$$\frac{C_A}{d_A} = \frac{C_B}{d_B}$$

For each circle, we will consider the k^{th} inscribed polygon as described in Section 14.1. We will denote these by the symbols $Poly_{k,A}$ and $Poly_{k,B}$. These polygons have perimeters $P_{k,A}$ and $P_{k,B}$. Each is regular polygon is made up of isosceles triangles.

In each of the isosceles triangles, the two congruent sides have length r_A and r_B.



The top angles have measure

$$\alpha = \frac{360}{3 \cdot 2^k}$$

Therefore, the isosceles triangles are similar, by the *SAS* Similarity Theorem. This gives us the equality of ratios

$$\frac{base_A}{base_B} = \frac{side_A}{side_B}$$

Cross multiplying, we obtain the new equation

$$\frac{base_A}{side_A} = \frac{base_B}{side_B}$$

But the bases of the isosceles triangles are the lengths $x_{k,A}$ and $x_{k,B}$ that are the lengths of the sides of the regular polygons $Poly_{k,A}$ and $Poly_{k,B}$. And the sides of the isosceles triangles are the radii of the circles, r_A and r_B. Substituting these symbols into our equation, we obtain

$$\frac{x_{k,A}}{r_A} = \frac{x_{k,B}}{r_B}$$

Here's a trick: Multiply each side of this equation by the constant $\frac{3 \cdot 2^k}{2}$. The result is the new equation

$$\frac{3 \cdot 2^k}{2} \cdot \frac{x_{k,A}}{r_A} = \frac{3 \cdot 2^k}{2} \cdot \frac{x_{k,B}}{r_B}$$

Rearranging factors, we obtain

$$\frac{3 \cdot 2^k \cdot x_{k,A}}{2r_A} = \frac{3 \cdot 2^k \cdot x_{k,B}}{2r_B}$$

We recognize the expressions for the perimiters of the polygons and for the diameters of the circles.

$$\frac{P_{k,A}}{d_A} = \frac{P_{k,B}}{d_B}$$

We take the limit of both sides of this equation as $k \to \infty$.

$$\lim_{k\to\infty}\left(\frac{P_{k,A}}{d_A}\right) = \lim_{k\to\infty}\left(\frac{P_{k,B}}{d_B}\right)$$

The denominators are constants, so the limits can be moved to the numerators.

$$\frac{\lim_{k\to\infty} P_{k,A}}{d_A} = \frac{\lim_{k\to\infty} P_{k,B}}{d_B}$$

Finally, notice that the values of the limits in the numerators will simply be the circumference of each circle!

$$\frac{C_A}{d_A} = \frac{C_B}{d_B}$$

End of proof

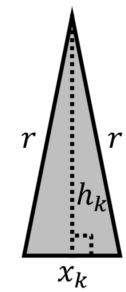
The theorem just presented makes precise the relationship between the circumference and diameter of a circle: their ratio is always the same real number. We give this real number a name.

Definition 105 The symbol pi, or π, denotes the real number that is the ratio $\frac{circumference}{diameter}$ for any circle. That is, $\pi = \frac{C}{d}$. (That this ratio is the same for all circles is guaranteed by Theorem 159, found on page 293.)

Rearranging the equation that defines π, we obtain the equation $C = \pi d$. We see that this equation is really just a restatement of the definition of π. It is not a formula that can be somehow proved. It is a definition of what the symbol π means.

From the results of the previous section, we can give a rough estimate of the value of π. For a circle of diameter $d = 1$, we have $C = \pi$. In the previous section, we found that for such a circle, the circumference C will be bounded by $3 < C \leq 4$. Therefore, $3 < \pi \leq 4$.

So far in this section, we have made precise the relationship between the circumference and diameter of a circle. They are related by the equation $C = \pi d$. Now let's make precise the relationship between the area and diameter, or the relationship between area and radius.



The k^{th} polygonal region $Poly_k$ can be subdivided into isosceles triangles like the one shown at right. The area of the triangular region is

$$\frac{1}{2} base \cdot height = \frac{1}{2} x_k \cdot h_k$$

We could come up with formulas for x_k and h_k, but it would be hard. Luckily, it will turn out that we don't need formulas for these lengths.

The polygonal region $Poly_k$ is made up of $n = 3 \cdot 2^k$ of the isosceles triangular regions, so its area will be

$$A_k = (3 \cdot 2^k) \cdot \left(\frac{1}{2} x_k \cdot h_k\right) = \frac{1}{2}(3 \cdot 2^k \cdot x_k) h_k$$

We recognize the expression in parentheses. It is the perimeter of the k^{th} polygonal region $Poly_k$. The perimeter is denoted by P_k.

$$A_k = \frac{1}{2}(3 \cdot 2^k \cdot x_k) h_k = \frac{1}{2} P_k h_k$$

The area of the circle will be the limit:

$$A = \lim_{k\to\infty} A_k = \lim_{k\to\infty} \left(\frac{1}{2} P_k h_k\right) = \frac{1}{2}\left(\lim_{k\to\infty} P_k\right)\left(\lim_{k\to\infty} h_k\right) = \frac{1}{2} Cr = \frac{\pi d r}{2} = \frac{\pi 2 r r}{2} = \pi r^2$$

There are a few things to notice here.

(1) The reason that we did not need formulas for x_k and h_k is that we only needed to consider their limits, and those limits were familiar.

(2) The formula for area involves the same constant π that shows up in the formula for circumference.

(3) The radius is squared in the formula for area. This does not surprise us, because from the section on similarity we are used to the idea that area is related to the square of lengths. But what *is* surprising is that the constant π is *not* squared in the formula for area.

Before going on, let's restate the formulas for circumference and area of a circle in the following theorem.

Theorem 160 The circumference of a circle is $c = \pi d$. The area of a circle is $A = \pi r^2$.

Proof

Definition 104 (found on page 291) states precisely what we mean by the circumference of a circle and the area of a circular region in terms of limits.

From Theorem 159 (found on page 293) we know that the ratio $\frac{circumference}{diameter}$ is the same for all circles. Definition 105 (found on page 294) introduces the symbol π as the value of that ratio. That is, $\pi = \frac{c}{d}$. This gives us the equation $c = \pi d$.

The calculations on the preceeding page gave us the formula $A = \pi r^2$.

End of Proof

With Theorem 160, we have answered the second question that was posed on page 289:

(2) How do we know that the formulas $c = \pi d$ and $A = \pi r^2$ are true?

Having made precise the relationships between circumference and area and the diameter and radius, we will now turn our attention to finding more precise estimates of the value of π.

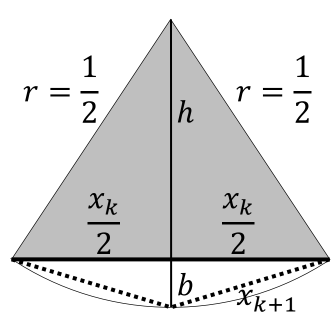
14.4. Approximations for *Pi*

From the formula $C = \pi d$, we know that π will be equal to the circumference of a circle of diameter $d = 1$. We defined that circumference in terms of a limit:

$$C = \lim_{k \to \infty} P_k$$

One approach to finding a value for C would be to find an explicit formula for P_k and then take the limit of that formula as $k \to \infty$. But an explicit expression is difficult to obtain. Instead, we will define P_k *recursively*. That is, we will show how to obtain a value for P_{k+1} from a known value of P_k. Since we know that $P_1 = 3$, we can use the recursive formula to make a list of values of P_k for $k = 1,2,3, \ldots$.

The k^{th} polygonal region $Poly_k$ can be subdivided into isosceles triangles like the shaded one shown at right. The solid horizontal segment that forms the base of the triangle is one of the sides of $Poly_k$. Its length is x_k, but for our purposes, it is more useful to break the segment into two segments of length $\frac{x_k}{2}$. The two congruent sides of the isosceles triangle have length $r = \frac{1}{2}$, the radius of the circle. The dotted segments are two sides of the $(k+1)^{st}$ polygonal region $Poly_{k+1}$. Each of the dotted segments has length x_{k+1}.



Our goal is to find the perimeter P_{k+1} in terms of P_k. We will start by finding the length x_{k+1} in terms of x_k.

$$
\begin{aligned}
x_{k+1} &= \sqrt{\left(\frac{x_k}{2}\right)^2 + b^2} \\
&= \sqrt{\left(\frac{x_k}{2}\right)^2 + \left(\frac{1}{2} - h\right)^2} \\
&= \sqrt{\left(\frac{x_k}{2}\right)^2 + \left(\frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{x_k}{2}\right)^2}\right)^2} \\
&= \frac{1}{2}\sqrt{{x_k}^2 + \left(1 - \sqrt{1 - {x_k}^2}\right)^2} \\
&= \frac{1}{2}\sqrt{{x_k}^2 + \left(1 - 2\sqrt{1 - {x_k}^2} + (1 - {x_k}^2)\right)} \\
&= \frac{1}{2}\sqrt{2 - 2\sqrt{1 - {x_k}^2}} \\
&= \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1 - {x_k}^2}}
\end{aligned}
$$

Now we use the fact that $P_k = 3 \cdot 2^k \cdot x_k$ so that $x_k = \frac{P_k}{3 \cdot 2^k}$

$$
x_{k+1} = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \left(\frac{P_k}{3 \cdot 2^k}\right)^2}}
$$

And finally, we use the fact that $P_{k+1} = 3 \cdot 2^{k+1} \cdot x_{k+1}$.

$$
\begin{aligned}
P_{k+1} &= \frac{3 \cdot 2^{k+1}}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \left(\frac{P_k}{3 \cdot 2^k}\right)^2}} \\
&= 3 \cdot 2^k \sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \left(\frac{P_k}{3 \cdot 2^k}\right)^2}}
\end{aligned}
$$

This is the equation that we were seeking. It shows us how to obtain a value for P_{k+1} from a known value of P_k. As an example of the use of this formula, we use the formula with $k = 1$ and $P_1 = 3$ and obtain

$$
P_2 = 3 \cdot 2^1 \sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \left(\frac{3}{3 \cdot 2^1}\right)^2}} = 6\sqrt{2 - \sqrt{3}} \approx 3.10582854
$$

Continuing, we can obtain a list of values of P_k for $k = 1,2,3, \ldots$. The values of P_k on this list will converge to a real number. The symbol for that real number is π.

How does the list look beyond the first two terms? The easiest way to answer that is to use the recursive formula for P_k to make a table of values in Excel.

k	n	P_k
1	6	3.00000000000000000000
2	12	3.10582854123025000000
3	24	3.13262861328124000000
4	48	3.13935020304687000000
5	96	3.14103195089053000000
6	192	3.14145247228534000000
7	384	3.14155760791162000000
8	768	3.14158389214894000000
9	1536	3.14159046323676000000
10	3072	3.14159210604305000000
11	6144	3.14159251658816000000
12	12288	3.14159261864079000000
13	24576	3.14159264532122000000
14	49152	3.14159264532122000000

The good news is that these numbers seem to be converging to a number close to 3.14159. The bad news is that from the 15^{th} decimal place on, all the digits are 0, and the that the values of P_{13} and P_{14} are exactly the same. This seems to indicate that the value of π is the number 3.14159264532122000000.

But this is suspicious. Consider the table entry for P_2. We found that the expression for P_2 was

$$P_2 = 6\sqrt{2 - \sqrt{3}}$$

In earlier courses, you learned that $\sqrt{3}$ is irrational. Therefore, $2 - \sqrt{3}$ will be irrational and $\sqrt{2 - \sqrt{3}}$ will be irrational and $P_2 = 6\sqrt{2 - \sqrt{3}}$ will be irrational.

The table says $P_2 = 3.10582854123025000000$. This is a rational number because it can be written

$$P_2 = 3.10582854123025000000 = \frac{310582854123025}{100000000000000}$$

So the value of P_2 given by Excel is definitely wrong. It should not end in zeroes.

Similar reasoning could be used to tell us that all of the other P_k values for $k \geq 2$ are also irrational, so the other P_k values in the table are wrong, as well. The problem is in round-off errors commited by Excel. A more scientific program like MATLAB could be used to produce a list with more digits of accuracy.

But what about the value of π? Is it irrational? We know that each P_k value for $k \geq 2$ is irrational, and that $\pi = \lim_{k\to\infty} P_k$, but that does not imply that π must be irrational. Indeed, it is possible to produce a sequence of irrational numbers that converges to a rational number. But in fact it has been shown that π is irrational.

In conclusion, we see that π is an irrational number close to 3.14159. So we have answered the first question that was posed on page 289:

(1) What is π?

Now we turn our attention to the concept of arc length.

14.5. Arc Length

The angle measure of an arc was defined in Definition 99, found on page 267 in Section 12.2. Recall that the angle measure is a number m in the range $0 < m < 360$. Now that we have seen a definition of the circumference of a circle, we are ready to discuss arc length.

We could start from scratch and consider segmented paths inscribed in circles, as we did at the beginning of this chapter. But that would be tedious and not terribly productive: our investigation of the circumference of a circle gave us enough of an idea of how that process works. Studying it again in a more difficult setting would not make us much wiser.

Instead, we will simply skip to the result of all that work and state it as a definition.

Definition 106 arc length

The length of an arc $\widehat{ABC}$ on a circle of radius r is defined to be the number

$$\hat{L}(\widehat{ABC}) = \frac{\hat{m}(\widehat{ABC})\pi r}{180}$$

.

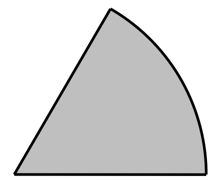
14.6. Area of Regions Bounded by Arcs and Line Segments

Now that we have defined the area of polygonal regions and circular regions it would be nice to have a notion of the area of regions bounded by arcs and line segments. In the case of polygonal regions, we first defined triangular regions, and then defined polygonal regions to be finite unions of triangular regions. In the case of regions bounded by arcs and line segments, it is not clear if there is a basic shape bounded by arcs and segments, a shape that can be arranged like pieces of a puzzle in order to make more general shapes bounded by arcs and segments.

For this edition of the book, we will not consider that general question. Instead, we will simply use our intuition to solve certain simple problems involving the area of regions bounded by arcs and line segments. Our main tools will be the following "rules" about area.

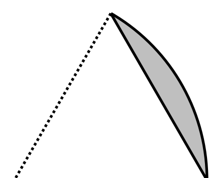
Definition 107 Rules for computing area

(1) The area of a triangular region is equal to one-half the base times the height. It does not matter which side of the triangle is chosen as the base.
(2) The area of a polygonal region is equal to the sum of the areas of the triangles in a complex for the region.
(3) The area of a circular region is πr^2.
(4) More generally, the area of a circular sector bounded by arc $\widehat{ABC}$ is $\pi r^2 \cdot \frac{\widehat{m}(\widehat{ABC})}{360}$.
(5) Congruence property: If two regions have congruent boundaries, then the area of the two regions is the same.
(6) Additivity property: If a region is the union of two smaller regions whose interiors do not intersect, then the area of the whole region is equal to the sum of the two smaller regions.



For example, at right is a slice of a pizza that had radius 1 foot. The crust is an arc of angle measure 60. Six of these slices arranged in a circle and touching without overlap would make up the entire pizza. By the additivity property, the sum of the areas of the six slices must equal the area of the pizza. By the congruence property, the areas of the six slices must be equal. Therefore, the area of this slice must be one sixth of the area of the pizza. That is, this slice must have area $A = \frac{\pi}{6}$ square feet. Another way of reaching the same result is to use rule (4) to compute the area. That is,

$$A = \pi r^2 \cdot \frac{\widehat{m}\left(\widehat{ABC}\right)}{360} = \pi(1)^2 \cdot \frac{60}{360} = \frac{\pi}{6}$$



At right is some leftover crust of the same pizza after an equilateral triangle has been eaten. The triangle has area $\frac{\sqrt{3}}{4}$ square feet, so by the additivity property, the leftover crust has area $A = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$ square feet.

In the exercises, you will see some more computational problems like this example involving adding and subtracting areas.

14.7. Exercises for Chapter 14

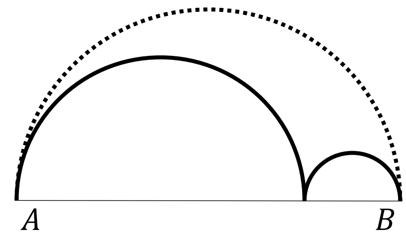
[1] In Section 14.4, we saw that an expression for P_2 was $P_2 = 6\sqrt{2-\sqrt{3}}$.
(A) Find the exact expression for P_3. Try to simplify it as much as I simplified the expression for P_2.
(B) Using a calculator, find a decimal approximation for P_3 that has more accurate digits than the decimal approximation presented in the table.

[2] I gave a pretty sketchy explanation of how we know that P_2 is irrational. See if you can explain it more thoroughly. One approach would be to do a sort of proof by contradiction. That is, assume that P_2 is rational. That means that it can be written as a ratio of integers,

$$6\sqrt{2-\sqrt{3}} = \frac{m}{n}$$

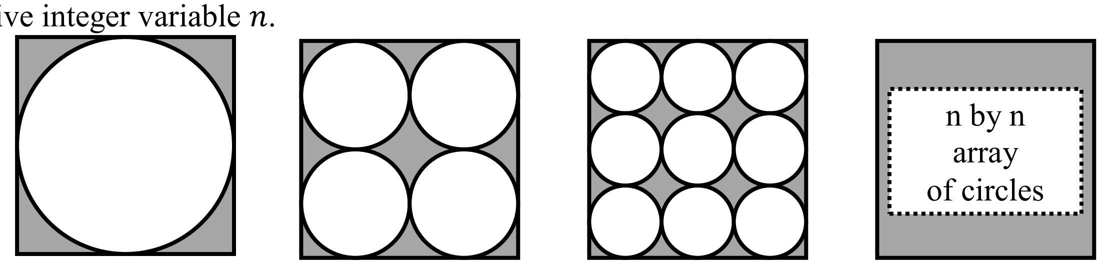
Square both sides of this expression. Do some arithmetic and reach a contradiction.

[3] Which path from point A to point B is longer: the one consisting of two small semicircles or the one consisting of one large semicircle? Justify your answer with calculations.



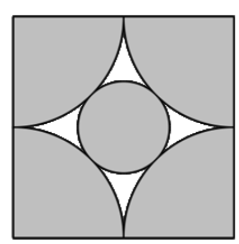
[4] OSHA has decided that a fence needs to be erected along the equator to keep people from falling from the northern hemisphere into the southern. The fence needs to have two rails: one that is three feet off the surface of the earth, and another that is low enough that Shemika Charles cannot limbo under it. What is the length of the top rail (in feet and inches), and what is the length of the bottom rail? Explain your calculations clearly. (You may assume that the earth is spherical, but you will need to look up its diameter.)

[5] In each picture below, the square has sides of length 1. Find the shaded areas. Show your work clearly. Note: You will see an obvious pattern in the answers to (a),(b),(c), so you will be able to guess the right answer to (d). But I want you to calculate the answer to (d) and show that it does indeed equal your guess. This will require that you figure out some way to work with the positive integer variable n.

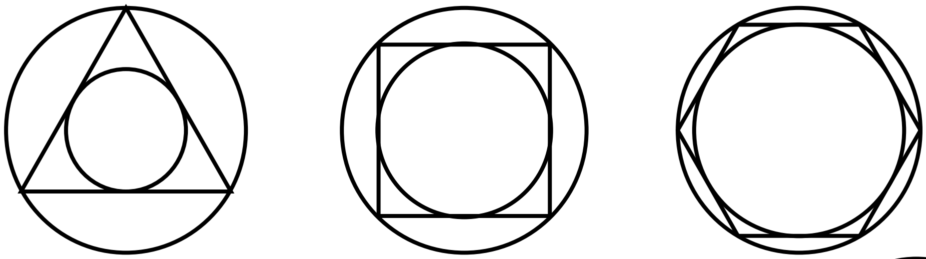


picture (a) picture (b) picture (c) picture (d)

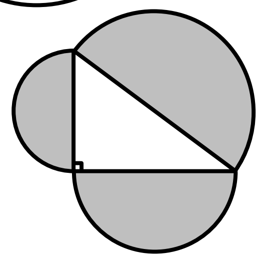
[6] In the figure at right, the outer square has sides of length 2. Find the value of the unshaded area. Show the details of the calculation clearly and simplify your answer.



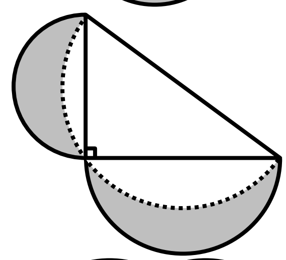
[7] In each picture below, a regular polygon is inscribed in the outer circle and is tangent to the inner circle. For each picture, find the ratio of the area of the outer circle to the area of the inner circle.



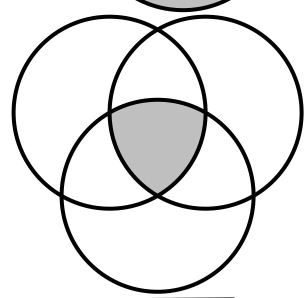
[8] Prove that the sum of the areas of the two smaller semicircular regions in the figure at right equals the area of the larger semicircular region.



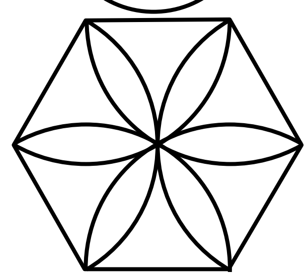
[9] In the figure at right, the two solid arcs are semicircles. The one large dotted arc is also a semicircle. Prove that the shaded area equals the area of the triangle. Hint: Use your result from [8].



[10] In the figure at right, each circle has the same radius r and passes through the center of the other two circles. Find the shaded area.



[11] In the figure at right, the hexagon has sides of length x. Find the area of the portion of the hexagon that is not covered up by the petals of the flower.



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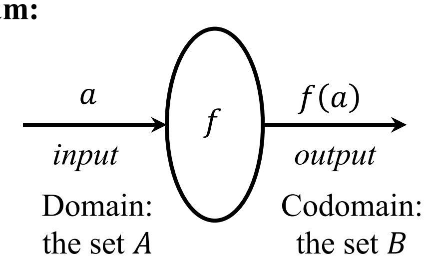
15. Maps, Transformations, Isometries

15.1. Functions

Recall our definition of function from Section 7.1:

Definition 51: function, domain, codomain, image, machine diagram; correspondence
Symbol: $f: A \to B$
Spoken: "f is a function that maps A to B."
Usage: A and B are sets. Set A is called the *domain* and set B is called the *codomain*.
Meaning: f is a machine that takes an element of set A as input and produces an element of set B as output.
More notation: If an element $a \in A$ is used as the input to the function , then the symbol $f(a)$ is used to denote the corresponding output. The output $f(a)$ is called *the image of* a *under the map* f.
Machine Diagram:



Additional notation: If f is both one-to-one and onto (that is, if f is a bijection), then the symbol $f: A \leftrightarrow B$ will be used. In this case, f is called a *correspondence* between the sets A and B.

When you studied functions in earlier courses, the domain and codomain were almost always sets of numbers. In Geometry, we often work with functions whose domains and codomains are sets of points. Even so, we will discuss many examples involving functions whose domain and codomain are sets of numbers, because they are simple and familiar.

Definition 108 Image and Preimage of a single element
If $f: A \to B$ and $a \in A$ is used as input to the function f, then the corresponding output $f(a) \in B$ is called the image of a.
If $f: A \to B$ and $b \in B$, then the preimage of b, denoted $f^{-1}(b)$, is the set of all elements of A whose image is b. That is, $f^{-1}(b) = \{a \in A \textit{ such that } f(a) = b\}$.

Observe that the image of a single element of the domain is a single element of the codomain, but the preimage of a single element of the codomain is a set.

For example, consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

- The image of 3 is $f(3) = 9$.
- The preimage of 9 is the set $f^{-1}(9) = \{-3,3\}$.
- The preimage of -5 is the empty set, because there is no real number a such that $a^2 = -5$.

Notice that in the definition of function, inputs are fed into the function one at a time. So in the symbol $f(a)$, the letter a represents a single element of the domain and the symbol $f(a)$ represents a single element of the codomain. In our study of functions in Geometry, we will often make use of the concept of the image of a set and the preimage of a set. Here is a definition.

Definition 109 Image of a Set and Preimage of a Set

If $f: A \to B$ and $S \subset A$, then the image of S, denoted $f(S)$, is the set of all elements of B that are images of elements of S. That is,

$$f(S) = \{b \in B \text{ such that } b = f(a) \text{ for some } a \in S\}$$

If $f: A \to B$ and $T \subset B$, then the preimage of T, denoted $f^{-1}(T)$, is the set of all elements of A whose images are elements of T. That is,

$$f^{-1}(T) = \{a \in A \text{ such that } f(a) \in T\}$$

For example, consider again the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

- The image of the set $\{-2,0,1,9\}$ is the set $f(\{-2,0,1,9\}) = \{0,1,4,81\}$.
- The preimage of the same set $\{-2,0,1,9\}$ is the set $f^{-1}(\{-2,0,1,9\}) = \{-3,-1,0,1,3\}$.

So with the new definition of the image of a set, we must keep in mind that when we encounter the symbol $f(thing)$, the $thing$ inside might be a single element of the domain—in which case the symbol $f(thing)$ represents a single element of the codomain—or the $thing$ inside might be a subset of the domain—in which case the symbol $f(thing)$ represents a subset of the codomain.

We will spend a lot of time studying the composition of functions. Here is the definition.

Definition 110 composition of fuctions, composite function

Symbol: $g \circ f$

Spoken: "g circle f",or "g after f", or "g composed with f"

Usage: $f: A \to B$ and $g: B \to C$

Meaning: the function $g \circ f: A \to C$ defined by $g \circ f(a) = g\big(f(a)\big)$.

Additional terminology: A function of the form $g \circ f$ is called a *composite function*.

For example, $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + 1$, we have the following two compostions:

(a) $g \circ f$ is the function defined by $g \circ f(x) = g\big(f(x)\big) = g(x^2) = x^2 + 1$.

(b) $f \circ g$ is the function defined by $f \circ g(x) = f\big(g(x)\big) = f(x + 1) = (x + 1)^2$.

Observe that in this example, $f \circ g \neq g \circ f$. This illustrates that function composition is usually *not commutative*. We will return to this terminology later in the chapter.

However, function composition is associative. That is, for all functions $f: A \to B$ and $g: B \to C$ and $h: C \to D$, the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal. To prove this, we need to show that when the two functions are given the same input, they always produce the same output. Here is the claim stated as a theorem. The proof follows.

Theorem 161 Function composition is associative.

For all functions $f: A \to B$ and $g: B \to C$ and $h: C \to D$, the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal.

Proof

For any $a \in A$, we simply compute the resulting outputs.

$$h \circ (g \circ f)(a) = h\big((g \circ f)(a)\big) = h\Big(g\big(f(a)\big)\Big)$$
$$(h \circ g) \circ f(a) = (h \circ g)\big(f(a)\big) = h\Big(g\big(f(a)\big)\Big)$$

Since the resulting outputs are the same, we conclude that the functions are the same.

End of Proof

In previous courses, you studied *one-to-one functions* and *onto functions*. Here are definitions

Definition 111 One-to-One Function

Words: The function $f: A \to B$ is *one-to-one*.
Alternate Words: The function $f: A \to B$ is *injective*.
Meaning in Words: Different inputs always produce different outputs.
Meaning in Symbols: $\forall x_1, x_2, if\ x_1 \neq x_2\ then\ f(x_1) \neq f(x_2)$.
Contrapositive: If two outputs are the same, then the inputs must have been the same.
Contrapositive in Symbols: $\forall x_1, x_2, if\ f(x_1) = f(x_2), then\ x_1 = x_2$.

Definition 112 Onto Function

Words: The function $f: A \to B$ is *onto*.
Alternate Words: The function $f: A \to B$ is *surjective*.
Meaning in Words: For every element of the codomain, there exists an element of the domain that will produce that element of the codomain as output.
Meaning in Symbols: $\forall y \in B, \exists x \in A\ such\ that\ f(x) = y$.

You may have heard the term *range* in previous courses. That word is problematic, because it has different meanings in different books.

- In some books, the *range* of a function is defined the way that we defined *codomain*. That is, for $f: A \to B$, the range of f would be defined as the set B.
- In some books, the *range* of a function is defined to be the image of the domain. That is, for $f: A \to B$, the range of f would be defined as the set $f(A) \subset B$.

The two uses of the words are not equivalent. For example, for the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, the codomain is the set of all real numbers, $\mathbb{R}$, while the image of the domain is the set of all non-negative real numbers. That is,

$$f(domain) = f(\mathbb{R}) = \mathbb{R}^{nonneg} = \{y \in \mathbb{R}\ such\ that\ y \geq 0\}$$

So the image of the domain is not the same set as the entire codomain in this case.

In this book, I have tried to avoid using terminology that has conflicting definitions in common use. For that reason, we will not use the word *range*. But I will sometimes refer to the image of the domain.

For example, it is useful to observe that the definition of an *onto* function could be summarized simply in the following way:

An *onto* function f is one for which $f(domain) = codomain$.

We will be interested in functions that are both one-to-one and onto. They have two commonly-used names:

Definition 113 Bijection, One-to-One Correspondence
Words: *"The function f is a bijection"*, or *"the function f is bijective"*.
Alternate Words: The function f is a *one-to-one correspondence*.
Meaning: The function f is both one-to-one and onto.

15.2. Inverse Functions

You might have learned in earlier courses that every bijective function has an inverse that is also a bijection. We will prove the fact here, because it gives us a good chance to review and because the proof is wonderfully simple if the notation is done right.

We start by abbreviating in symbols the definition of function (Definition 51). That definition says that a function $f: A \to B$ takes any element of the domain A as input and produces as output exactly one element of the codomain B. In the language of quantifiers, this would be written as follows:

Definition of function, using quantifiers:
For every $a \in A$, there exists a unique $b \in B$ such that $f(a) = b$.

In symbols, we use the exclamation point "!" as the abbreviation for the word *unique* and the colon ":" as the abbreviation for the words *such that*. Using those symbols, the definition of function would be abbreviated as follows:

Definition of function, using quantifiers, abbreviated in symbols:
$\forall a \in A, \exists!\, b \in B: f(a) = b$

Now let's abbreviate in symbols the meaning of the word *bijection*. A bijection is both *one-to-one* and *onto*. The definition of *onto* says

Definition of *onto*, using quantifiers:
For every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

For the present discussion, it helps to be clearer about this: the words *there exists* really mean there *exists at least one*.

Definition of *onto*, using quantifiers, clarified
For every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

But a bijection f is both *onto* and *one-to-one*. The fact that f is *one-to-one* means that there cannot be more than one $a \in A$ such that $f(a) = b$. Combined with the fact that f is *onto*, we see that there exists *exactly one* $a \in A$ such that $f(a) = b$. In other words, there exists a *unique* $a \in A$ such that $f(a) = b$.

> Definition of bijection, using quantifiers:
> For every $b \in B$, there exists a *unique* $a \in A$ such that $f(a) = b$.

This is ready to be abbreviated:

> Definition of bijection, using quantifiers, abbreviated in symbols:
> $\forall b \in B, \exists!\, a \in A\colon f(a) = b$

Summarizing, here is the definition of a bijective function, written in symbols.

> Definition of bijective function $f\colon A \to B$ in symbols:
> (i) $\forall a \in A, \exists!\, b \in B\colon f(a) = b$ (definition of function)
> (ii) $\forall b \in B, \exists!\, a \in A\colon f(a) = b$ (definition of bijective)

Now consider defining a new symbol as follows:

> **New Symbol:** $g(b) = a$.
> **Meaning:** $f(a) = b$.

It might seem silly to do that. But now consider what happens if we substitue the new symbol into the two symbolic expressions above. The result is two new symbolic expressions

> (new i) $\forall a \in A, \exists!\, b \in B\colon g(b) = a$
> (new ii) $\forall b \in B, \exists!\, a \in A\colon g(b) = a$

Observe that expression (new ii) tells us that g is qualified to be called a function, $g\colon B \to A$. And observe that expression (new i) tells us that the function g is a bijection!

Let's explore the function g further. In particular, let's study what happens when we compose f and g. Note that because $f\colon A \to B$ and $g\colon B \to A$, we can compose them in either order. That is,

> The function $g \circ f$ will be a function with domain A and codomain A. That is, $g \circ f\colon A \to A$.
> The function $f \circ g$ will be a function with domain B and codomain B. That is, $f \circ g\colon B \to B$.

Consider what happens when we feed inputs into these composite functions. For this discussion, suppose that $f(a) = b$. Then $g(b) = a$ where $a \in A$ and $b \in B$.

Start with the function $g \circ f$. Given $a \in A$ as input, what will be the output $g \circ f(a)$?

$$\begin{aligned} g \circ f(a) &= g\big(f(a)\big) && \text{meaning of the composition symbol} \\ &= g(b) && \text{because } f(a) = b \\ &= a && \text{because } g(b) = a \end{aligned}$$

We see that when $a \in A$ is used as input to the function $g \circ f$, the output will be $g \circ f(a) = a$.

Now consider the function $f \circ g$. Given some $b \in B$ as input, what will be the output $f \circ g(b)$?

$$\begin{aligned} f \circ g(b) &= f(g(b)) && \text{meaning of the composition symbol} \\ &= f(a) && \text{because } g(b) = a \\ &= b && \text{because } f(a) = b \end{aligned}$$

We see that when $b \in B$ is used as input to the function $f \circ g$, the output will be $f \circ g(b) = b$.

It is worthwhile to stop here and summarize what we have done.

Given a bijective function $f: A \to B$, we defined a new symbol $g(b) = a$ to mean the same thing as the symbol $f(a) = b$. We saw that this defines a bijective function $g: B \to A$. The compositions of functions f and g have the following two properties:

$$\forall a \in A, g \circ f(a) = a$$
$$\forall b \in B, f \circ g(b) = b$$

In math, functions f and g that have the two properties above are called *inverses* of one another. The two properties are called *inverse relations*. Here is the definition:

Definition 114 Inverse Functions, Inverse Relations

Words: Functions f and g are *inverses* of one another.

Usage: $f: A \to B$ and $g: B \to A$

Meaning: f and g satisfy the following two properties, called inverse relations:

$$\forall a \in A, g \circ f(a) = a$$
$$\forall b \in B, f \circ g(b) = b$$

Additional Symbols and Terminology: Another way of saying that functions f and g are inverses of one another is to say that g is the inverse of f. Instead of using different letters for a function and its inverse, it is common to use the symbol f^{-1} to denote the inverse of a function f. With this notation, we would say that $f: A \to B$ and $f^{-1}: B \to A$, and the inverse relations become:

$$\forall a \in A, f^{-1} \circ f(a) = a$$
$$\forall b \in B, f \circ f^{-1}(b) = b$$

Our discussion on the previous two pages, in which we introduced the function g and discussed its properties, can now be summarized using the terminology of inverse functions. It is worth presenting this as a theorem.

Theorem 162 Bijective functions have inverse functions that are also bijective.

If $f: A \to B$ is a bijective function, then f has an inverse function $f^{-1}: B \to A$. The inverse function is also bijective.

Proof: The discussion on the previous few pages.

Another theorem from earlier courses is the following:

Theorem 163 If a function has an inverse function, then both the function and its inverse are bijective.
If functions $f: A \to B$ and $g: B \to A$ are inverses of one another (that is, if they satisfy the inverse relations), then both f and g are bijective.

The proof of Theorem 163 is easy if one makes use of the following wonderful trick. The trick can be employed any time it is known that a function $f: A \to B$ has an inverse function. Recall that the meaning of the term *inverse function* is that f and f^{-1} satisfy the inverse relations

$$\forall a \in A, f^{-1} \circ f(a) = a$$
$$\forall b \in B, f \circ f^{-1}(b) = b$$

Here's the trick. Because we know the inverse relations are satisfied, any element $a \in A$ can be replaced by the expression $f^{-1}(f(a))$. That may seem silly, because the expression $f^{-1}(f(a))$ takes up a lot more room on the page than the expression a. But there are times (such as in the proof of Theorem 163) when the longer expression is useful. Similarly, any element $b \in B$ can be replaced by the expression $f(f^{-1}(b))$. And of course, the trick works the other way as well. That is, the expression $f^{-1}(f(a))$ can be replaced by the letter a, and the expression $f(f^{-1}(b))$ can be replaced by the letter b. But this doesn't seem so tricky.

Now we will prove Theorem 163, making use of the trick.

Proof of Theorem 163
Part 1: Show that f is one-to-one and that g is one-to-one.
(1) Suppose that $f(a_1) = f(a_2)$.
(2) Then $g(f(a_1)) = g(f(a_2))$ because g is a function.
(3) Therefore, $a_1 = a_2$, because g is the inverse of f. (Here we have used the trick.)

Steps (1) – (3) prove that f is one-to-one. Analogous steps would prove that g is also one-to-one.
End of Proof Part 1

Part 2: Show that f is onto and that g is onto.

To show that $f: A \to B$ is onto, we must show that for any $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

(7) Suppose that $b \in B$.
(8) We can write $b = f(g(b))$, because f and g are inverses of each other. (Here we have used the trick.)
(9) Observe that $g(b)$ is an element of the set A, because $g: B \to A$. So let $a = g(b)$. Then $b = f(a)$. We have found an $a \in A$ such that $f(a) = b$.

Steps (7) – (9) prove that f is onto. Analogous steps would prove that g is also onto.
End of Proof Part 2

Theorem 163 is useful for proving that a function is bijective without having to do any work. That sounds very vague, so let me give an example to illustrate. Here is a theorem about the inverse of a composition of functions. The proof will use Theorem 163.

Theorem 164 about the inverse of a composition of functions

If functions $f: A \to B$ and $g: B \to C$ are both bijective, then their composition $g \circ f$ will be bijective. The inverse of the composition will be $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof

Part 1: Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Consider the composition of $(g \circ f)$ and $(f^{-1} \circ g^{-1})$. Observe that

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ \left(f \circ (f^{-1} \circ g^{-1})\right) \\ &= g \circ \left((f \circ f^{-1}) \circ g^{-1}\right) \\ &= g \circ (id \circ g^{-1}) \\ &= g \circ g^{-1} \\ &= id\end{aligned}$$

Similarly,

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = id$$

Since the functions $(g \circ f)$ and $(f^{-1} \circ g^{-1})$ satisfy the inverse relations, we conclude they are inverses of one another. Or, we could say that $(f^{-1} \circ g^{-1})$ is the inverse of $(g \circ f)$. That is, $(f^{-1} \circ g^{-1}) = (g \circ f)^{-1}$.

Part 2: Prove that $g \circ f$ is bijective.

We have found an inverse function for $g \circ f$. By Theorem 163, $g \circ f$ must be bijective.

End of Proof

15.3. Maps of the Plane

As mentioned at the start of this chapter, in Geometry we will be interested in functions whose domain and codomain are sets of points. Recall from Definition 20 that the symbol $\mathcal{P}$ is used to denote the set of all points. So far in our course, we have not used any other name for that set. But many books refer to the set $\mathcal{P}$ as *the plane*. We can make it official with a definition.

Definition 115 The *plane* is defined to be the set $\mathcal{P}$ of all points.

With this terminology, we can refer to maps of the plane.

Definition 116 A *map of the plane* is defined to be a function $f: \mathcal{P} \to \mathcal{P}$.

We will be interested in maps of the plane that have certain other properties. In particular, we will be interested in maps of the plane that are bijective. We give such functions the name *tranformations*. Here is the official definition.

Definition 117 A *transformation of the plane* is defined to be a bijective map of the plane. The set of all transformations of the plane is denoted by the symbol T.

But we are more interested in maps of the plane that are *distance preserving*. We will give such functions the name *isometries*. Here is the official definition.

Definition 118 Isometry of the Plane

Words: f is an isometry of the plane.

Meaning: f is a distance preserving map of the plane. That is, for all points P and Q, the distance from P to Q is the same as the distance from $f(P)$ to $f(Q)$.

Meaning in symbols: $\forall P, Q \in \mathcal{P}, d(P,Q) = d\big(f(P), f(Q)\big)$.

In the next section, we will study transformations of the plane. In later sections, we will study isometries of the plane.

15.4. Transformations of the Plane

We will start this section with a quick presentation of three examples of Transformations of the Plane.

Our first example of a transformation of the plane is so simple that it might seem silly to introduce it. But in fact, it is a very important function.

Definition 119 The Identity Map of the Plane is the map $id: \mathcal{P} \to \mathcal{P}$ defined by $id(Q) = Q$ for every point Q.

Observe that the identity map of the plane is clearly both one-to-one and onto, so it is a transformation of the plane.

This is a good time to introduce the terminology of *fixed points*. Here is a definition.

Definition 120 a Fixed Point of a Map of the Plane

Words: Q is a fixed point of the map f.

Meaning: $f(Q) = Q$

We see that every point in the plane is a fixed point of the identity map of the plane.

Our second example of a transformation of the plane is the dilation. Here is the definition.

Definition 121 The Dilation of the Plane

Symbol: $D_{C,k}$

Spoken: The dilation centered at C with scaling factor k

Usage: C is a point, called the *center* of the dilation, and k is a positive real number.

Meaning: The map $D_{C,k}: \mathcal{P} \to \mathcal{P}$ defined as follows

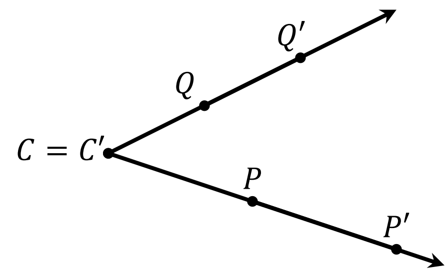
The point C is a fixed point of $D_{C,k}$. That is, $D_{C,k}(C) = C$.

When a point $Q \neq C$ is used as input to the map $D_{C,k}$, the output is the unique point $Q' = D_{C,k}(Q)$ that has these two properties:

- Point Q' lies on ray $\overrightarrow{CQ}$
- The distance $d(C, Q') = kd(C, Q)$

(The existence and uniqueness of such a point Q' is guaranteed by the Congruent Segment Construction Theorem, (Theorem 24).)

For example, consider the dilation with $k = 2$, denoted by the symbol $D_{C,2}$. It has C as a fixed point. If any other point $Q \neq C$ is used as input to the map $D_{C,2}$, the output will be the point Q' as shown. in the figure at right.



It is probably reasonably clear to you that the dilation $D_{C,k}$ is one-to-one and onto. That is, it is a transformation of the plane. You are asked to provide the details in an exercise.

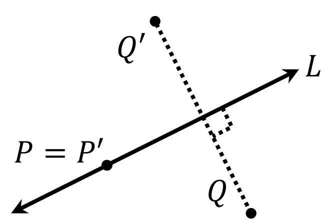
Our third example of a transformation of the plane is the reflection. Here is the definition.

Definition 122 The Reflection of the Plane

Symbol: M_L
Spoken: The reflection in line L
Usage: L is a line, called the *line of reflection*
Meaning: The map $M_L : \mathcal{P} \to \mathcal{P}$ defined as follows

Every point on the line L is a fixed point of M_L. That is, if $P \in L$ then $M_L(P) = P$.
When a point Q not on line L is used as input to the map M_L, the output is the unique point $Q' = M_L(Q)$ such that line L is the perpendicular bisector of segment $\overline{QQ'}$.
(The existence and uniqueness of such a point Q' is can be proven using the axioms and theorems of Neutral Geometry. You are asked to provide the details in an exercise.)

For example, see the reflection M_L with line of reflection L as shown in the figure at right.



It is probably reasonably clear to you that the reflection M_L is one-to-one and onto. That is, it is a transformation of the plane. You are asked to provide the details in an exercise.

15.5. Review of Binary Operations and Groups

In the next section, we will show that the set of transformations of the plane is a group. We should start by reviewing the definition of a group. That is what we will do in this section. In order to do that, we need to first review the definition of a binary operation on a set.

Definition 123 A binary operation on a set S is a function $* : S \times S \to S$.

In other words, a binary operation is a function that takes as input a pair of elements of the set S and produces as output a single element of the set S.

The definition of a binary operation might be unfamiliar to you, but in fact you have been working with binary operations since grade school.

Example #1 of a binary operation:

The operation of addition is a binary operation on the set of integers. That is, addition may be thought of as a function, $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. If a pair of integers is used as input to the function, the resulting output is an integer. For example, $+(2,5) = 7$. But of course, this is not the way that we are used to writing the addition operation. We would write $2 + 5 = 7$ instead. But it is the same operation.

Example #2 of a binary operation:

The operation of subtraction is a binary operation on the set of integers. That is, subtraction may be thought of as a function, $-: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. If a pair of integers is used as input to the function, the resulting output is an integer. For example, $-(2,5) = -3$. As with the operation of addition, this is not the way that we are used to writing the subtraction operation. We would write $2 - 5 = -3$ instead. But it is the same operation.

Example #3 of a binary operation:

The operation of multiplication is a binary operation on the set of real numbers. That is, multiplication may be thought of as a function, $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. If a pair of real numbers is used as input to the function, the resulting output is a real number. For example, $*\left(\pi, \frac{1}{2}\right) = \frac{\pi}{2}$. Again, this is not the way that we are used to writing the multiplication operation. We would write $\pi * \frac{1}{2} = \frac{\pi}{2}$ instead. But it is the same operation.

The three examples just presented illustrate a common notation for binary operations. The symbol that represents the operation is often placed in between the elements in the pair that is the input to the operation instead of in front of the pair. For example, we write $2 + 5 = 7$ instead of writing $+(2,5) = 7$.

We will be interested in four properties that a binary operation may or may not have. They are *associativity*, the existence of an *identity*, the existence of *inverses*, and *commutativity*. Here are the definitions and examples.

We start with the definition of the *associativity* property.

Definition 124 associativity, associative binary operation

Words: "$*$ is associative" or "$*$ has the associativity property"

Usage: $*$ is a binary operation $*$ on some set S.

Meaning: $\forall a, b, c \in S, a * (b * c) = (a * b) * c$

For example, addition on the set of integers (Example #1 above) is an associative binary operation. (Associativity of addition is specified in the axioms for integer arithmetic.)

But subtraction on the set of integers (Example #2 above) is not associative. As a counterexample, observe that

$$10 - (8 - 5) \neq (10 - 8) - 5$$

So subtraction is a binary operation, but it is not associative.

The second property that we will discuss is the existence of an *identity* element.

Definition 125 identity element, binary operation with an identity element
Words: "$*$ has an identity element."
Usage: $*$ is a binary operation $*$ on some set S.
Meaning:.There is an element $\exists e \in S$ with the following property:
$$\forall a \in S, a * e = e * a = a$$
Meaning in symbols: $\exists e \in S: \forall a \in S, a * e = e * a = a$
Additional Terminology: The element $\exists e \in S$ is called *the identity for operation* $*$.

For example, addition on the set of integers (Example #1 above) has an identity element: the integer 0. We say that 0 is *the additive identity element*.

As a second example, multiplication on the set of integers has an identity element: the integer 1. We say that 1 is *the multiplicative identity element*.

As a third example, consider the operation of multiplication on the set of even integers. This is a binary operation, because when two even integers are multiplied, the result is an even integer. But there is no identity element. That is because the only integer that could possibly be an identity element would be the integer 1, but 1 is not an even integer.

As a fourth example, consider the binary operation of subtraction on the set of integers. (Example #2 above). Notice that the number 0 has the property that for any integer m, the equation $m - 0 = m$ is true. Based on this, one might suspect that the number 0 might be an identity element for the operation of subtraction. But notice that the equation $0 - m = m$ is not always true. For example, $0 - 0 = 0$ is true, but $0 - 5 = 5$ is false. Since the equation $0 - m = m$ is not always true, the integer 0 is not qualified to be called an identity element for the operation of subtraction. It should be clear that the operation of subtraction on the set of integers does not have an identity element.

The example just presented shows the significance of the expression $a * e = e * a = a$ in the definition of an identity element. That single expression means that the equalities $a * e = a$ and $e * a = a$ must both be satisfied. It is sometimes possible to find an element that satisfies one of the equalities but not both. Such an element is not qualified to be called an identity element.

The third property that we will discuss is the existence of an *inverse* for each element.

Definition 126 binary operation with inverses
Words: "$*$ has inverses."

Usage: $*$ is a binary operation $*$ on some set S.
Meaning: For each element $a \in S$, there exists is an $a^{-1} \in S$ such that

$$a * a^{-1} = e \text{ and } a^{-1} * a = e.$$

Meaning in symbols: $\forall a \in S, \exists a^{-1} \in S: a * a^{-1} = a^{-1} * a = e$
Additional Terminology: The element $a^{-1} \in S$ is called the *inverse* of a.

For example, consider the binary operation of addition on the set of real numbers. For this operation, the set of real numbers contians an inverse for each element. For the real number x, the inverse is the real number $-x$. We would say that $-x$ is the *additive inverse* of x.

For another example, consider the binary operation of multiplication on the set of real numbers. For this operation, the set of real numbers contians an inverse for some elements, but not for every element. If $x \neq 0$, then the real number $\frac{1}{x}$ is a multiplicative inverse for x. But the real number 0 does not have a multiplicative inverse. So we must say that the operation of multiplication on the set of real numbers does not have an inverse for every element.

The notation of inverse elements can be confusing. For an element a, the generic symbol for the inverse element is a^{-1}. But for different binary operations, different symbols are sometimes used. Maybe a table will help clarify.

binary operation	element	generic symbol for the inverse of the element	common symbol for the inverse of the element
addition on $\mathbb{R}$	x	x^{-1} (never used in this context)	$-x$ the additive inverse of x
multiplication on $\mathbb{R}$	x	x^{-1}	x^{-1} or $\frac{1}{x}$ the multiplicative inverse of x

The commutative property is our fourth and final definition of a property that binary operations may or may not have.

Definition 127 commutativity, commutative binary operation
Words: "$*$ is commutative" or "$*$ has the commutative property"
Usage: $*$ is a binary operation $*$ on some set S.
Meaning: $\forall a, b, c \in S, a * b = b * a$

For example, addition on the set of integers (Example #1 above) is a commutative binary operation. (Commutativity of addition is specified in the axioms for integer arithmetic.)

But subtraction on the set of integers (Example #2 above) is a non-commutative binary operation. As a counterexample, observe that $10 - 7 \neq 7 - 10$.

Now that we have introduced the four properties *associativity*, the existence of an *identity*, the existence of *inverses*, and *commutativity*, we are ready to indiscuss the definition of a group.

Definition 128 Group

A Group is a pair $(G,*)$ consisting of a set G and a binary operation $*$ on G that has the following three properties.

(1) Associativity (Definition 124)
(2) Existence of an Identity Element (Definition 125)
(3) Existence of an Inverse for each Element (Definition 126)

Notice that the definition of group does not include any mention of commutativity, the fourth property that we introduced above. Some groups will have this property; some will not. There is a special name for those groups that do have the property.

Definition 129 Commutative Group, Abelian Group

A commutative group (or abelian group) is a group $(G,*)$ that has the commutativity property (Definition 127)

For instance, consider the binary operation of addition on the set of integers.

(1) Note that or all integers a, b, c, the equation $a + (b + c) = (a + b) + c$ is true. So the operation is associative.
(2) Consider the integer 0. Observe that for all integers m, the equations $m + 0 = m$ and $0 + m = m$ are both true. Therefore, the integer 0 is qualified to be called an identity element for the operation of addition.
(3) For any integer m, observe that the number $-m$ is an integer and that the two equations $m + (-m) = 0$ and $(-m) + m = 0$ are both true. So the integer $-m$ is qualified to be called an additive inverse for the integer m.

We conclude that the pair $(\mathbb{Z}, +)$ is a group. Observe that it is a commutative group, because it also has the fourth important property:

Commutative property: for all integers m, n the equation $m + n = n + m$ is true.

On the other hand, consider the binary operation of multiplication on the set of integers.

(1) Note that or all integers a, b, c, the equation $a * (b * c) = (a * b) * c$ is true. So the operation is associative.
(2) Consider the integer 1. Observe that for all integers m, the equations $m * 1 = m$ and $1 * m = m$ are both true. Therefore, the integer 1 is qualified to be called an identity element for the operation of multiplication.
(3) The integer 5 does not have a multiplicative inverse. The real number $\frac{1}{5}$ does have the property that $\frac{1}{5} * 5 = 1$ and $5 * \frac{1}{5} = 1$, but the real number $\frac{1}{5}$ is not an integer.

We conclude that the pair $(\mathbb{Z},*)$ is not group because the set $\mathbb{Z}$ does not contain a multiplicitive inverse for each element.

15.6. The Set of Transformations of the Plane is a Group

In this short section, we will prove the following theorem.

Theorem 165 the pair $(T,\circ)$ consisting of the set of Transformations of the Plane and the operation of composition of functions, is a group.

Proof of the theorem

Part (0): Prove that $\circ$ is a binary operation on the set T.

To prove that $\circ$ is a binary operation on the set T, we must prove that if f and g are elements of T, then $f \circ g$ is also an element of T. That is, we must show that if f and g are bijective maps of the plane, then $f \circ g$ is also bijective. But this was proven in Theorem 164.

Part 1: Prove that the binary operation $\circ$ on the set T is associative.

In Theorem 161 we proved that function composition is associative. Transformations of the plane are just a particular kind of function. Therefore composition of transformations is associative.

Part 2: Prove that there exists an identity element in the set T.

This is easy. Consider the map $id: \mathcal{P} \to \mathcal{P}$ introduced in Definition 119. We have observed that it is both one-to-one and onto, so it is a transformation of the plane. We must consider its composition with other transformations. In particular, we must show that for any transformation f, the equations $id \circ f = f$ and $f \circ id = f$ are both true. Realize that these are equations about equality of functions. To prove that two functions are equal, one must prove that when fed the same input, they produce the same ouput.

So we must consider the output for any given point $Q \in \mathcal{P}$. Observe that

$$id \circ f(Q) = id\big(f(Q)\big) = f(Q)$$

This tells us that the functions $id \circ f$ and f are the same function. Therefore, the equation $id \circ f = f$ is true.

A similar calculation would show that the equation $f \circ id = f$ is true.

Part 3: Prove that there is an inverse for each element.

Transformations of the plane are bijections. Theorem 162 tells us that bijective functions have inverses that are also bijections. So every transformation of the plane has an inverse that is also a transformation of the plane. This all sounds good, but we need to be careful. In the context of Theorem 162, the inverse of a function f iss a function called f^{-1} that has the following property:

For every $Q \in \mathcal{P}$, the equations $f^{-1} \circ f(Q) = Q$ and $f \circ f^{-1}(Q) = Q$ are both true.

In our present context, the inverse of a function f is a function called f^{-1} that has this property:

$$f^{-1} \circ f = id \text{ and } f \circ f^{-1} = id$$

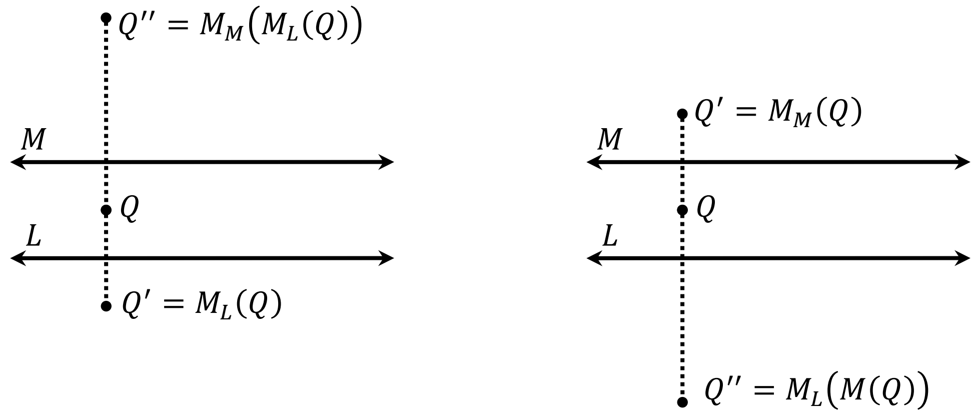
We see that the two properties mean the same thing. That is, the kind of inverse that is guaranteed by Theorem 162 is equivalent to the kind of inverse that we need for a binary operation.

End of Proof

We have proved that the set of transformations of the plane is a group, that is, that the set of transformations has the first three important properties of binary operations that we introduced in the previous section. So it is natural to wonder if the set of transformations also has the fourth property, commutativity. That is, is the set of transformations an abelian group? It is very easy to find a counterexample that shows that the group is non-abelian. Here's one:

Example to illustrate that the group of Transformations of the Plane is non-abelain.

Let M_K and M_L be the reflections in the lines K and L shown in the figures below. For the given point Q, the outputs $Q'' = M_K \circ M_L(Q)$ and $Q'' = M_L \circ M_K(Q)$ are shown. We see that the two outputs are not the same. Therefore, the transformations $M_K \circ M_L$ and $M_L \circ M_K$ are not the same.



15.7. Isometries of the Plane

In the previoius sections, we studied maps of the plane that are also bijective, the so-called *transformations of the plane*. But as mentioned in Section 15.3, we are more interested in the *isometries of the plane*. Those are the distance preserving maps of the plane. In this section, we will begin our study of isometries of the plane by showing that they have a number of other properties.

For starters, it is very easy to prove that the composition of two isometries is another isometry:

Theorem 166 The composition of two isometries of the plane is also an isometry of the plane.
Proof

Suppose that f and g are isometries of the plane.
Let P, Q be any two points. Then

$$\begin{aligned} d(g \circ f(P), g \circ f(Q)) &= d\left(g(f(P)), g(f(Q))\right) \quad (definition\ of\ composition) \\ &= d(f(P), f(Q)) \quad (because\ g\ preserves\ distance) \\ &= d(P, Q) \quad (because\ f\ preserves\ distance) \end{aligned}$$

So $g \circ f$ preserves distance. That is, $g \circ f$ is an isometry.
End of proof

It is also very easy to prove that every isometry of the plane is also *one-to-one*. Here is the theorem and a quick proof:

Theorem 167 Every isometry of the plane is one-to-one.

Proof

Suppose that f is an isometry of the plane and that P and Q are two points such that $f(P) = f(Q)$. That is, the two symbols $f(P)$ and $f(Q)$ represent the same point. (We must show that P and Q are in fact the same point.)

$$\begin{aligned} 0 &= d(f(P), f(P))\ because\ f(P) = f(Q) \\ &= d(P, Q)\ because\ f\ is\ distance\ preserving \end{aligned}$$

Therefore, P and Q are the same point.
End of proof

It is a little harder to prove that every isometry is also *onto*. One goal for the rest of this section will be to prove that fact. But we will start by proving that isometries also preserve collinearity.

Our first theorem is really just a restatement of facts that have been proven in two earlier theorems. We restate them here in a form that is useful for the current section.

Theorem 168 For three distinct points, betweenness is related to distance between the points.
For distinct points A, B, C, the following two statements are equivalent.
(i) $A * B * C$
(ii) $d(A, B) + d(B, C) = d(A, C)$

Proof
(1) Suppose that A, B, C are distinct points.
(2) Either they are collinear or they are not.

Case I: Points $\boldsymbol{A, B, C}$ are non-collinear.
(3) Suppose that points A, B, C are non-collinear.
(4) Then the statement (i) is false, because part of the definition of the symbol $A * B * C$ is that the three points are collinear.
(5) $d(A, B) + d(B, C) > d(A, C)$ by Theorem 64, the Triangle Inequality, applied to the three non-collinear points A, B, C. So statement (ii) is false.
(6) We see that in this case, statements (i) and (ii) are both false.

Case II: Points $\boldsymbol{A, B, C}$ are collinear.
(7) Suppose that points A, B, C are collinear.
(8) Then statements (i) and (ii) are equivalent by Theorem 16

Conclusion
(9) We see that in either case, statements (i) and (ii) are equivalent.
End of proof

Because we have seen that betweenness of points is related to the distances between the points, it should come as no surprise that isometries of the plane preserve collinearity. Here is the theorem and a quick proof.

Theorem 169 Isometries of the plane preserve collinearity.
If A, B, C are distinct, collinear points and f is an isometry of the plane, then $f(A), f(B), f(C)$ are distinct, collinear points.

Proof
(1) Suppose that A, B, C are distinct, collinear points.
(2) Then $f(A), f(B), f(C)$ are distinct points (because f is one-to-one, by Theorem 167)
(3) Exactly one of the three points A, B, C is between the other two (by Theorem 15). Assume that it is point B that is the one in the middle, so $A * B * C$. (If not, rename the three points so that point B *is* the one in the middle.)
(4) Then $d(A, B) + d(B, C) = d(A, C)$ (by (3) and Theorem 168 (i)➔(ii)).
(5) So $d\big(f(A), f(B)\big) + d\big(f(B), f(C)\big) = d\big(f(A), f(C)\big)$. (because f preserves distance).
(6) Therefore, $f(A) * f(B) * f(C)$. (by (2), (5), and Theorem 168 (ii)➔(i)).
End of proof

Now we can prove that every isometry of the plane is *onto*.

Theorem 170 Every isometry of the plane is onto.

Proof

(1) Suppose f is an isometry of the plane.
(2) Let Y be any point. (We must show that there exists a point X such that $f(X) = Y$.)
(3) There exist two distinct points A, B.
(4) There are three possibilities for $f(A)$ and $f(B)$ and Y.
  (i) $f(A) = Y$ or $f(B) = Y$.
  (ii) $f(A) \neq Y$ and $f(B) \neq Y$ but $f(A)$ and $f(B)$ and Y are collinear.
  (iii) $f(A)$ and $f(B)$ and Y are non-collinear.

Case (i)

(5) If $f(A) = Y$ or $f(B) = Y$ then simply let $X = A$ or $X = B$. We have shown that in this case, there exists a point X such that $f(X) = Y$.

Case (ii)

(6) Suppose that $f(A) \neq Y$ and $f(B) \neq Y$ but $f(A)$ and $f(B)$ and Y are collinear.

Introduce point $\boldsymbol{X}$.

(7) Let L be line $\overleftrightarrow{AB}$ and let M be line $\overleftrightarrow{f(A)f(B)}$.
(8) Let X be the point on line L such that $d(X,A) = d\big(Y, f(A)\big)$ and $d(X,B) = d\big(Y, f(B)\big)$. (It is not hard to show that such a point X exists, but it is tedious. It can be done by using coordinate systems on lines L and M, or by using the congruent segment construction theorem.)

Show that $\boldsymbol{f(X) = Y}$.

(9) Observe that A, B, X are distinct, collinear points. (Although we do not know which one is in the middle.)
(10) By Theorem 169, we know that $f(A), f(B), f(X)$ are also distinct collinear points. So point $f(X)$ lies on line M.
(11) Because f preserves distance, we know that $d\big(f(X), f(A)\big) = d(X,A) = d\big(Y, f(A)\big)$ and that $d\big(f(X), f(B)\big) = d(X,B) = d\big(Y, f(B)\big)$. This tells us that points $f(X)$ and Y must be the same point on line M.
(12) We have shown that in this case, there exists a point X such that $f(X) = Y$.

Case (iii)

(13) Suppose that $f(A)$ and $f(B)$ and Y are non-collinear.

Introduce point $\boldsymbol{Z}$.

(14) Let $r_1 = d\big(Y, f(A)\big)$ and $r_2 = d\big(Y, f(B)\big)$. Then point Y is an intersection point of the two circles $Circle(f(A), r_1)$ and $Circle(f(B), r_2)$.
(15) Because $f(A)$ and $f(B)$ and Y are non-collinear, we know that the two circles will also intersect at another point that we can call Z.

Introduce points $\boldsymbol{W}, \boldsymbol{X}$.

(16) Observe that the points Y, Z are the same distances from point $f(A)$ and are the same distances from point $f(B)$. That is,

$$d\big(Y, f(A)\big) = d\big(Z, f(A)\big) = r_1$$

$$d(Y, f(B)) = d(Z, f(B)) = r_2$$

(17) Because $d(A, B) = d(f(A), f(B))$, we know that the two circles $Circle(A, r_1)$ and $Circle(B, r_1)$ will also intersect at two points that we can call W and X.

Show that $f(\{W, X\}) = \{Y, Z\}$.

(18) Observe that the points W, X are the same distances from point A and are the same distances from point B. That is,

$$d(W, A) = d(X, A) = r_1$$
$$d(W, B) = d(X, B) = r_2$$

(19) Because f is distance preserving, we know

$$d(f(W), f(A)) = d(W, A) = r_1$$
$$d(f(X), f(A)) = d(X, A) = r_1$$
$$d(f(W), f(B)) = d(W, B) = r_2$$
$$d(f(X), f(B)) = d(X, B) = r_2$$

In other words, points $f(W), f(X)$ must lie at the two intersection points of the two circles $Circle(f(A), r_1)$ and $Circle(f(B), r_2)$. But those two intersection points are Y and Z. This could be written in symbols as $f(\{W, X\}) = \{Y, Z\}$.

(20) So the output points $f(W), f(X)$ must be the same points as Y, Z, although we don't know which one is which. But f is one-to-one, so we do know that the two inputs do not both produce the same output. Therefore, one of the inputs will definitely produce an output of Y. We can interchange the names of W, X if necessary so that $f(X) = Y$.

(21) We have shown that in this case, there exists a point X such that $f(X) = Y$.

Conclusion

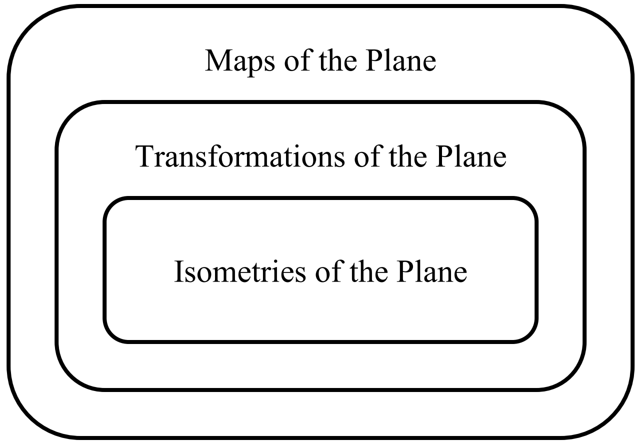
(11) We have shown that in every case, there exists a point X such that $f(X) = Y$.

End of Proof

The following corollary follows immediately from Theorem 167 and Theorem 170 and Definition 117.

Theorem 171 (corollary) Every isometry of the plane is also a transformation of the plane.

The Venn diagram at right summarizes the relationship between the types of maps of the plane that we have studied so far.



Note that because every isometry of the plane is a transformation of the plane, we know that every isometry of the plane will have an inverse. This fact could be stated as a quick corollary of the theorem just proved. But instead, I will combine it with another fact as part of the following theorem.

Theorem 172 Every isometry of the plane has an inverse that is also an isometry.

Proof

(1) Suppose that f is an isometry of the plane.
(2) Then f is a transformation of the plane **(Justify)** and so it has an inverse f^{-1} that is also a transformation of the plane. **(Justify)** That is, both f and f^{-1} are transformations of the plane.
(3) Let P, Q be any two points. Then

$$\begin{aligned} d\big(f^{-1}(P), f^{-1}(Q)\big) &= d\Big(f\big(f^{-1}(P)\big), f\big(f^{-1}(Q)\big)\Big) \ \ (because\ f\ is\ an\ isometry) \\ &= d(P,Q) \ \ (inverse\ relations) \end{aligned}$$

(4) So f^{-1} preserves distance. That is, f^{-1} is also an isometry.

End of Proof

15.8. The set of isometries of the plane is a group

In Section 15.6, we proved that the set of transformations of the plane, with the operation of composition, is a group (Theorem 165). In this section, we will prove the same for the set of isometries.

Theorem 173 the pair $(I, \circ)$ consisting of the set of Isometries of the Plane and the operation of composition of functions, is a group.

Proof of the theorem

Part (0): Prove that $\circ$ is a binary operation on the set *I*.

To prove that $\circ$ is a binary operation on the set I, we must prove that if f and g are elements of I, then $f \circ g$ is also an element of I. That is, we must show that if f and g are isometries of the plane, then $f \circ g$ is also an isometry of the plane. But this was proven in Theorem 166.

Part 1: Prove that the binary operation $\circ$ on the set *I* is associative.

In Theorem 161 we proved that function composition is associative. Isometries of the plane are just a particular kind of function. Therefore composition of isometries is associative.

Part 2: Prove that there exists an identity element in the set *I*.

The map $id: \mathcal{P} \to \mathcal{P}$ introduced in Definition 119 is an isometry of the plane. It is also an identity map in the sense of composition of maps of the plane. That is, for any f that is a map of the plane, the equations $f \circ id = f$ and $id \circ f = f$ are both true.

Part 3: Prove that there is an inverse for each element.

Theorem 172 tells us that every isometry of the plane has an inverse that is alsO an isometry. In the context of Theorem 172, the inverse of a function f is a function called f^{-1} that has the following property:

For every $Q \in \mathcal{P}$, the equations $f^{-1} \circ f(Q) = Q$ and $f \circ f^{-1}(Q) = Q$ are both true.

In our present context, the inverse of a function f is a function called f^{-1} that has this property:

$$f^{-1} \circ f = id \text{ and } f \circ f^{-1} = id$$

We see that the two properties mean the same thing. That is, the kind of inverse that is guaranteed by Theorem 172 is equivalent to the kind of inverse that we need for a binary operation.

End of Proof

15.9. Some More Properties of Isometries

Theorem 174 Isometries of the plane preserve lines.

If L is a line and $f: \mathcal{P} \to \mathcal{P}$ is an isometry, then the image $f(L)$ is also a line.

Proof

(1) Suppose that L is a line and $f: \mathcal{P} \to \mathcal{P}$ is an isometry.
(2) There exist two distinct points A, B on line L. (Justify.)
(3) $f(A) \neq f(B)$. (Justify.)
(4) There is a unique line passing through points $f(A)$ and $f(B)$. (Justify.) Call this line M.

Our goal now is to prove that $f(L) = M$. This is a statement about equality of two sets. To prove it, we must prove that $f(L) \subset M$ and that $M \subset f(L)$.

Part 1: Prove that $\boldsymbol{f(L) \subset M}$.

(5) To prove that $f(L) \subset M$, we must show that if $P \in L$ then $f(P) \in M$.
(6) In the case the $P = A$ or $P = B$, we already know that $f(P) \in M$ because of the way that we defined line M.
(7) So suppose that A, B, P are distinct points on L.
(8) Then $f(A), f(B), f(P)$ are distinct, collinear points. (Justify.) So point $f(P)$ lies on line M.

Part 2: Prove that $\boldsymbol{M \subset f(L)}$.

To prove that $M \subset f(L)$, we must show that if $Q \in M$ then there exists a point $P \in L$ such that $f(P) = Q$.

(9) In the case the $Q = f(A)$ or $Q = f(B)$, then we're done: we can let $P = A$ or $P = B$.
(10) So suppose that $f(A), f(B), Q$ are distinct points on M.
(11) The isometry f has an inverse f^{-1} that is also an isometry. (Justify.)

(12) Then $f^{-1}(f(A)), f^{-1}(f(B)), f^{-1}(Q)$ are distinct, collinear points. (Justify.) So point $f(P)$ lies on line M. That is, $A, B, f^{-1}(Q)$ are distinct, collinear points. (Justify.) So point $f^{-1}(Q)$ lies on line L.

(13) Let $P = f^{-1}(Q)$. Then P lies on L and $f(P) = f(f^{-1}(Q)) = Q$.

Conclusion of cases

(14) We see that in either case, there exists a point $P \in L$ such that $f(P) = Q$.

Conclusion of Proof

(15) We have proven that $f(L) \subset M$ and that $M \subset f(L)$. Conclude that $f(L) = M$.

End of Proof

Here is an analogous theorem for circles.

Theorem 175 Isometries of the plane preserve circles.

If $f: \mathcal{P} \to \mathcal{P}$ is an isometry, then the image of a circle is a circle with the same radius. More specifically, the image $f(Circle(P,r))$ is the circle $Circle(f(P),r)$.

Proof

(1) Suppose that $Circle(P,r)$ is given and that $f: \mathcal{P} \to \mathcal{P}$ preserves distance.

Our goal is to prove that $f(Circle(P,r)) = Circle(f(P),r)$. This is a statement about equality of sets. To prove it, we must prove that $f(Circle(P,r)) \subset Circle(f(P),r)$ and that $Circle(f(P),r) \subset f(Circle(P,r))$.

Part 1: Show that $\boldsymbol{f(Circle(P,r)) \subset Circle(f(P),r)}$.

To show that $f(Circle(P,r)) \subset Circle(f(P),r)$, we must show that if $X \in Circle(P,r)$, then $f(X) \in Circle(f(P),r)$.

(2) Suppose that $X \in Circle(P,r)$.

(3) Then

$$\begin{aligned} r &= d(X,P) \quad (\textit{because } X \textit{ is on the circle}) \\ &= d(f(X), f(P)) \quad (\textit{because } f \textit{ preserves distance}) \end{aligned}$$

(4) So $f(X) \in Circle(f(P),r)$. (by (3))

End of Proof Part 1

Part 2: Show that $\boldsymbol{Circle(f(P),r) \subset f(Circle(P,r))}$.

To show that $Circle(f(P),r) \subset f(Circle(P,r))$, we must show that if $Y \in Circle(f(P),r)$, then there exists a point $X \in Circle(P,r)$ such that $f(X) = Y$.

(5) Suppose that $Y \in Circle(f(P),r)$.

(6) The isometry f has an inverse f^{-1} that is also an isometry. (Justify.)

(7) Observe that

$$\begin{aligned} r &= d(y, f(p)) \ (\textit{because } Y \textit{ is on } Circle(f(P),r)) \\ &= d\left(f^{-1}(Y), f^{-1}(f(P))\right) \ (\textit{because } f^{-1} \textit{ is an isometry}) \\ &= d(f^{-1}(Y), P) \ (\textit{inverse relations}) \end{aligned}$$

(8) So point $f^{-1}(Y)$ lies on $Circle(P,r)$. Let $X = f^{-1}(Y)$. Then X lies on $Circle(P,r)$ and $f(X) = f\big(f^{-1}(Y)\big) = Y$.
(9) We have shown that for every $Y \in Circle(f(P),r)$, there exists a point $X \in Circle(P,r)$ such that $f(X) = Y$. This shows that $Circle(f(P),r) \subset f\big(Circle(P,r)\big)$.

End of Proof Part 2

Conclusion of Proof

(10) We have shown that $f\big(Circle(P,r)\big) \subset Circle(f(P),r)$ and that $Circle(f(P),r) \subset f\big(Circle(P,r)\big)$. This proves that $f\big(Circle(P,r)\big) = Circle(f(P),r)$.

End of Proof

It turns out that isometries of the plane can be completely determined by the outputs that they produce when three non-collinear points are used as inputs. The following theorem is about the situation where the three non-collinear input points are fixed points.

Theorem 176 If an isometry has three non-collinear fixed points, then the isometry is the identity map.

Proof

(1) Suppose that $f: \mathcal{P} \to \mathcal{P}$ is an isometry with three non-collinear fixed points A, B, C.
(2) Let Q be any point. (We must show that $f(Q) = Q$.)
(3) Assume that $f(Q) \neq Q$.
(4) Note that Q is not any of the points A, B, C, because they are all fixed points.
(5) Observe that

$$\begin{aligned} d(A,Q) &= d\big(f(A), f(Q)\big) \quad (because\ f\ is\ an\ isometry) \\ &= d\big(A, f(Q)\big) \quad (because\ A\ is\ a\ fixed\ point) \end{aligned}$$

That is, point A is equidistant from points Q and $f(Q)$.
(6) Therefore, A lies on the line that is the perpendicular bisector of segment $\overline{Qf(Q)}$.
(7) Similarly, we could show that point B is equidistant from points Q and $f(Q)$, so B also lies on the line that is the perpendicular bisector of segment $\overline{Qf(Q)}$.
(8) And we could show that point C is equidistant from points Q and $f(Q)$, so C lies on the line that is the perpendicular bisector of segment $\overline{Qf(Q)}$.
(9) We have shown that points A, B, C all lie on the line that is the perpendicular bisector of segment $\overline{Qf(Q)}$.
(10) Statement (9) contradicts statement (1) that says that points A, B, C are non-collinear. Therefore, our assumption in step (3) was wrong. It must be that $f(Q) = Q$. In other words, the map f is the identity map.

End of Proof

The theorem just proved is the heart of the proof of the following theorem that essentially says that an isometry is completely determined by its image at three non-collinear fixed points.

Note that the proof uses the trick described after Theorem 163.

Theorem 177 If two isometries have the same images at three non-collinear fixed points, then the isometries are in fact the same isometry.

If $f: \mathcal{P} \to \mathcal{P}$ and $g: \mathcal{P} \to \mathcal{P}$ are isometries and A, B, C are non-collinear points such that $f(A) = g(A)$ and $f(B) = g(B)$ and $f(C) = g(C)$, then $f = g$.

Proof

(1) Suppose that $f: \mathcal{P} \to \mathcal{P}$ and $g: \mathcal{P} \to \mathcal{P}$ are isometries and A, B, C are non-collinear points such that $f(A) = g(A)$ and $f(B) = g(B)$ and $f(C) = g(C)$.
(2) Let Q be any point. (We must show that $f(Q) = g(Q)$.)
(3) The map g has an inverse g^{-1} that is also an isometry. (Justify.)
(4) The map $g^{-1} \circ f$ is also an isometry. (Justify.)
(5) Observe that

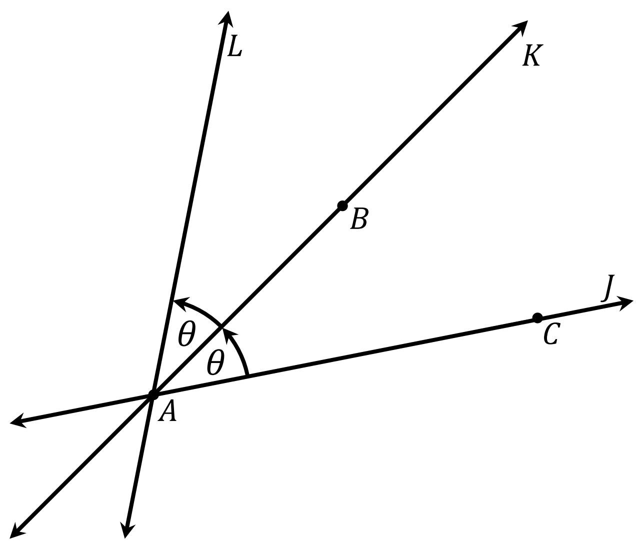
$$\begin{aligned} g^{-1}\big(f(A)\big) &= g^{-1}\big(g(A)\big) \quad \big(\textit{because } f(A) = g(A)\big) \\ &= A \quad (\textit{inverse relation}) \end{aligned}$$

That is, point A is a fixed point of the isometry $g^{-1} \circ f$.
(6) Similarly, we could show that point B is a fixed point of the isometry $g^{-1} \circ f$.
(8) And we could show that point C is a fixed point of the isometry $g^{-1} \circ f$.
(9) We have shown that the non-collinear points A, B, C are all fixed points of the isometry $g^{-1} \circ f$. Therefore, $g^{-1} \circ f = id$. (Justify.)
(10) Similarly, we could introduce the the isometry $f \circ g^{-1}$ and show that points A, B, C are all fixed point of it. Therefore, $f \circ g^{-1} = id$. (Justify.)
(11) Statements (9) and (10) tell us that g^{-1} must be the inverse of f. But g^{-1} is the inverse of g. So $f = g$.

End of Proof

Here is a simple example of the use of Theorem 177.

For the lines J, K, L shown, prove that $M_L \circ M_K \circ M_J = M_K$.



Solution: For each map $M_L \circ M_K \circ M_J$ and M_K, we will consider the images of the non-collinear points A, B, C.

(i) Describe the trajectories of the three points A, B, C under the mapping

$$M_L \circ M_K \circ M_J$$

The trajectory of point A under the map $M_L \circ M_K \circ M_J$ is:

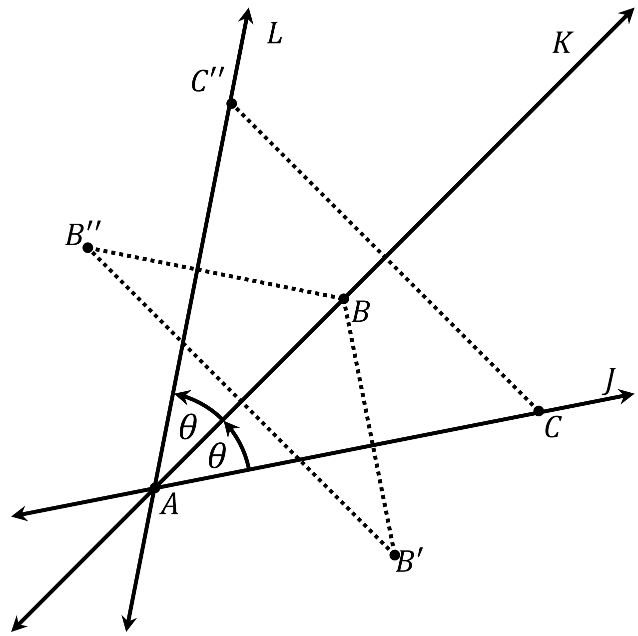
$$A \to A \to A \to A$$

The trajectory of point B under the map $M_L \circ M_K \circ M_J$ is:

$$B \to B' \to B'' \to B$$

The trajectory of point C under the map $M_L \circ M_K \circ M_J$ is:

$$C \to C \to C'' \to C''$$



(ii) Now Describe the trajectories of the three points A, B, C under the mapping M_K

The trajectory of point A under the map M_K is:

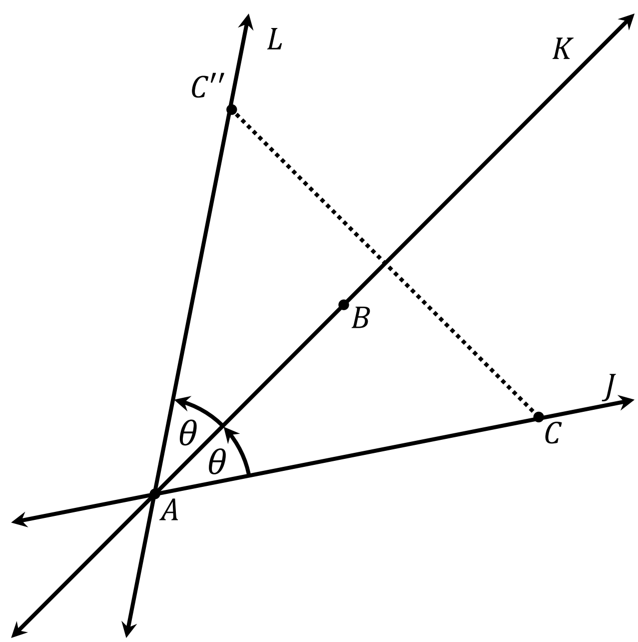
$$A \to A$$

The trajectory of point B under the map M_K is:

$$B \to B$$

The trajectory of point C under the map M_K is:

$$C \to C''$$



(iii) Conclusion: We see that

$$M_L \circ M_K \circ M_J(A) = M_K(A)$$
$$M_L \circ M_K \circ M_J(B) = M_K(B)$$
$$M_L \circ M_K \circ M_J(C) = M_K(C)$$

Therefore, $M_L \circ M_K \circ M_J = M_K$ by Theorem 177.

15.10. Exercises

[1] (a) Prove that the dilation map $D_{C,k}$ (introduced in Definition 121) is bijective.
(b) What is ${D_{C,k}}^{-1}$? Explain.

[2] (a) Prove the existence and uniqueness of the point Q' described in the definition of M_L, the reflection in line L (introduced in Definition 122).

(b) Prove that the reflection M_L is bijective.

(c) What is ${M_L}^{-1}$? Explain.

[3] Prove that the reflection M_L is an isometry.

[4] Prove that the dilation $D_{C,k}$ is not an isometry.

[5] Justify the steps in the proof of Theorem 172, which states that every isometry of the plane has an inverse that is also an isometry.

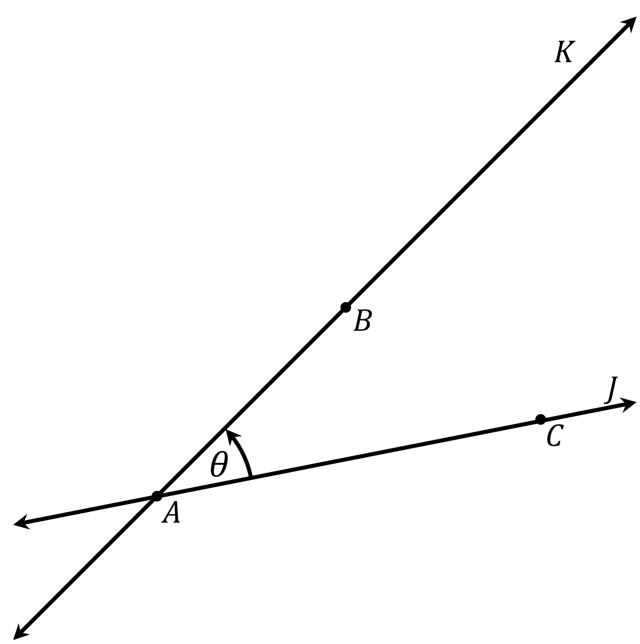
[6] In the proof of Theorem 176 (If an isometry has three non-collinear fixed points, then the isometry is the identity map.), step (3) is the assumption that $f(Q) \neq Q$. This fact does not get mentioned explicitly later in the proof. Explain where this fact is used.

[7] For the lines J, K and the non-collinear points A, B, C shown,

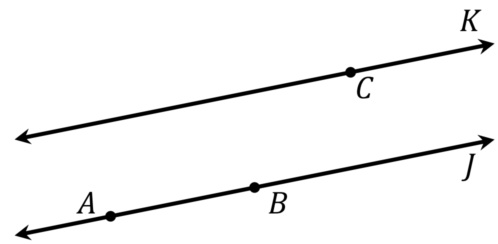
(a) Find the images of the three points A, B, C under the mapping $M_K \circ M_J$. That is,

Find $M_K \circ M_J(A)$.
Find $M_K \circ M_J(B)$.
Find $M_K \circ M_J(C)$.

(b) Based on your answer to (A), what is another name for the mapping $M_K \circ M_J$? Explain. (You might need to make up some notation. Go ahead.)



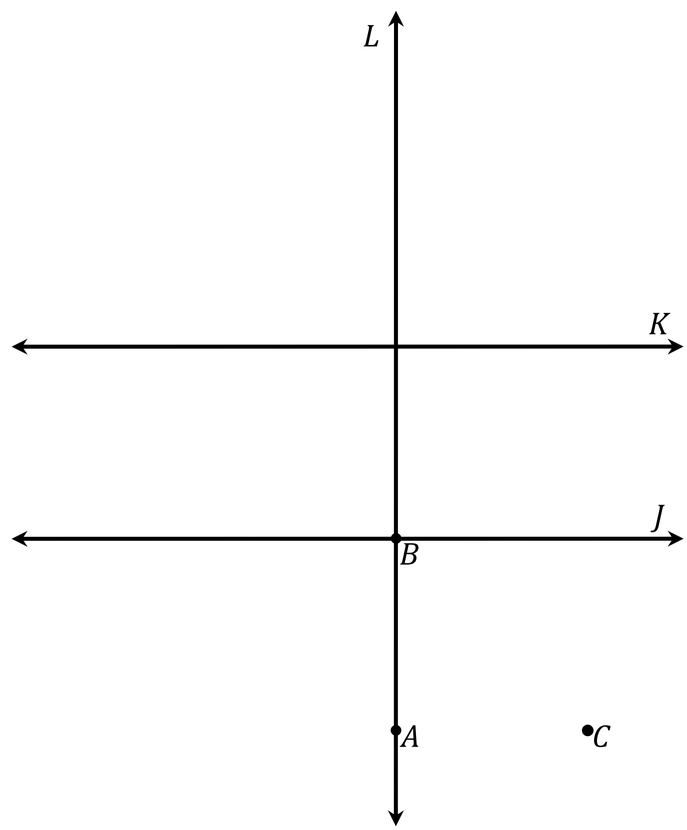
[8] For the parallel lines J, K and the non-collinear points A, B, C shown,



(a) Find the images of the three points A, B, C under the mapping $M_K \circ M_J$. That is, find $M_K \circ M_J(A)$ and find $M_K \circ M_J(B)$ and find $M_K \circ M_J(C)$.

(b) Based on your answer to (A), what is another name for the mapping $M_K \circ M_J$? Explain.

[9] For the lines J, K, L and the non-collinear points A, B, C shown,



(a) Find the images of the three points A, B, C under the mapping

$$M_L \circ M_K \circ M_J$$

That is, find

$$M_L \circ M_K \circ M_J(A)$$

and find

$$M_L \circ M_K \circ M_J(B)$$

and find

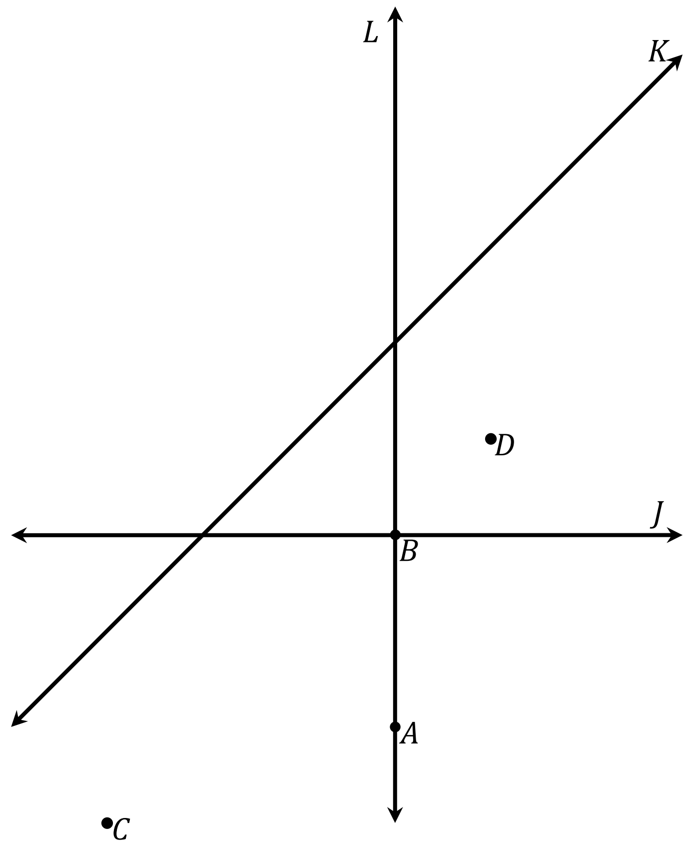
$$M_L \circ M_K \circ M_J(C).$$

(b) Based on your answer to (A), what is another name for the mapping

$$M_L \circ M_K \circ M_J?$$

Explain.

[10] For the lines J, K, L and the non-collinear points A, B, C, D shown,



(a) Find the images of the four points A, B, C, D under the mapping

$$M_L \circ M_K \circ M_J$$

That is, find

$$M_L \circ M_K \circ M_J(A)$$

and find

$$M_L \circ M_K \circ M_J(B)$$

and find

$$M_L \circ M_K \circ M_J(C)$$

and find

$$M_L \circ M_K \circ M_J(D).$$

(b) Based on your answer to (A), what is another name for the mapping

$$M_L \circ M_K \circ M_J?$$

Explain.

Appendix 1: List of Definitions

Definition 1: Interpretation of an axiom system (page 18)
Suppose that an axiom system consists of the following four things

- an undefined object of one type, and a set A containing all of the objects of that type
- an undefined object of another type, and a set B containing all of the objects of that type
- an undefined relation $\mathcal{R}$ from set A to set B
- a list of axioms involving the primitive objects and the relation

An interpretation of the axiom systems is the following three things

- a designation of an actual set A' that will play the role of set A
- a designation of an actual set B' that will play the role of set B
- a designation of an actual relation $\mathcal{R}'$ from A' to B' that will play the role of the relation $\mathcal{R}$

Definition 2: successful interpretation of an axiom system; model of an axiom system (page 20)
To say that an interpretation of an axiom system is *successful* means that when the undefined terms and undefined relations in the axioms are replaced with the corresponding terms and relations of the interpretation, the resulting statements are all true. A *model* of an axiom system is an *interpretation* that is *successful*.

Definition 3: isomorphic models of an axiom system (page 21)
Two models of an axiom system are said to be *isomorphic* if it is possible to describe a correspondence between the objects and relations of one model and the objects and relations of the other model in a way that all corresponding relationships are preserved.

Definition 4: consistent axiom system (page 21)
An axiom system is said to be *consistent* if it is possible for all of the axioms to be true. The axiom system is said to be *inconsistent* if it is not possible for all of the axioms to be true.

Definition 5: dependent and independent axioms (page 24)
An axiom is said to be *dependent* if it is possible to prove that the axiom is true as a consequence of the other axioms. An axiom is said to be *independent* if it is not possible to prove that it is true as a consequence of the other axioms.

Definition 6: independent axiom system (page 26)
An axiom system is said to be *independent* if all of its axioms are independent. An axiom system is said to be *not independent* if one or more of its axioms are not independent.

Definition 7: complete axiom system (page 26)
An axiom system is said to be *complete* if any two models of the axiom system are isomorphic. An axiom system is said to be *not complete* if there exist two models that are not isomorphic.

Definition 8: Alternate definition of a complete axiom system (page 28)

An axiom system is said to be *not complete* if it *is possible* to write an *additonal independent statement* regarding the primitive terms and relations. (An *additional independent statement* is a statement *S* that is *not* one of the axioms and such that there is a model for the axiom system in which Statement *S* is true and there is also a model for the axiom system in which Statement *S* is false.) An axiom system is said to be *complete* if it *is not possible* to write such an additional independent statement.

Definition 9: passes through (page 35)

- words: Line *L passes through* point *P*.
- meaning: Point *P* lies on line *L*.

Definition 10: intersecting lines (page 35)

- words: Line *L intersects* line *M*.
- meaning: There exists a point (at least one point) that lies on both lines.

Definition 11: parallel lines (page 35)

- words: Line *L* is parallel to line *M*.
- symbol: $L \| M$.
- meaning: Line *L* does not intersect line *M*. That is, there is no point that lies on both lines.

Definition 12: collinear points (page 35)

- words: The set of points $\{P_1, P_2, \ldots, P_k\}$ is collinear.
- meaning: There exists a line *L* that passes through all the points.

Definition 13: concurrent lines (page 36)

- words: The set of lines $\{L_1, L_2, \ldots, L_k\}$ is concurrent.
- meaning: There exists a point *P* that lies on all the lines.

Definition 14: Abstract Model, Concrete Model, Relative Consistency, Absolute Consistency (page 51)

- An abstract model of an axiom system is a model that is, itself, another axiom system.
- A concrete model of an axiom system is a model that uses actual objects and relations.
- An axiom system is called relatively consistent if an abstract model has been demonstrated.
- An axiom system is called absolutely consistent if a concrete model has been demonstrated.

Definition 15: the concept of *duality* and the *dual* of an axiomatic geometry (page 52)

Given any axiomatic geometry with primitive objects *point* and *line*, primitive relation "*the point lies on the line*", and defined relation "*the line passes through the point*", one can obtain a new axiomatic geometry by making the following replacements.

- Replace every occurrence of *point* in the original with *line* in the new axiom system.
- Replace every occurrence of *line* in the original with *point* in the new axiom system.
- Replace every occurrence of *lies on* in the original with *passes through* in the new.
- Replace every occurrence of *passes through* in the original with *lies on* in the new.

The resulting new axiomatic geometry is called the *dual* of the original geometry. The dual geometry will have primitive objects *line* and *point*, primitive relation "*the line passes through the point*", and defined relation "*the point lies on the line*." Any theorem of the original axiom system can be translated as well, and the result will be a valid theorem of the new dual axiom system.

Definition 16: self-dual geometry (page 56)

An axiomatic geometry is said to be *self-dual* if the statements of the dual axioms are true statements in the original geometry.

Definition 17: The Axiom System for Neutral Geometry (page 61)

Primitive Objects: *point*, *line*

Primitive Relation: *the point lies on the line*

Axioms of Incidence and Distance

<N1> There exist two distinct points. (at least two)

<N2> For every pair of distinct points, there exists exactly one line that both points lie on.

<N3> For every line, there exists a point that does not lie on the line. (at least one)

<N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Axiom of Separation

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following three properties:

(i) The three sets L, H_1, H_2 form a partition of the set of all points.

(ii) Each of the half-planes is convex.

(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

Axioms of Angle measurement

<N7> (Angle Measurement Axiom) There exists a function $m: \mathcal{A} \to (0{,}180)$, called the Angle Measurement Function.

<N8> (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $m(\angle PAB) = r$.

<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Axiom of Triangle Congruence

<N10> (*SAS* Axiom) If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Definition 18: the unique line passing through two distinct points (page 63)

words: line P, Q

symbol: $\overleftrightarrow{PQ}$

usage: P and Q are distinct points

meaning: the unique line that passes through both P and Q. (The existence and uniqueness of such a line is guaranteed by Axiom <N2>.)

Definition 19: The set of all abstract points is denoted by the symbol $\mathcal{P}$ and is called *the plane*. (page 69)

Definition 20: abbreviated symbol for the distance between two points (page 69)
abbreviated symbol: PQ
meaning: the distance between points P and Q, that is, $d(P,Q)$

Definition 21: Coordinate Function (page 70)
Words: f is a coordinate function on line L.
Meaning: f is a function with domain L and codomain $\mathbb{R}$ (that is, $f: L \to \mathbb{R}$) that has the following properties:
(1) f is a one-to-one correspondence. That is, f is both one-to-one and onto.
(2) f "agrees with" the distance function d in the following way:
For all points P,Q on line L, the equation $|f(P) - f(Q)| = d(P,Q)$ is true.
Additional Terminology: In standard function notation, the symbol $f(P)$ denotes the output of the coordinate function f when the point P is used as input. Note that $f(P)$ is a real number. The number $f(P)$ is called *the coordinate of point P on line L.*
Additional Notation: Because a coordinate function is tied to a particular line, it might be a good idea to have a notation for the coordinate function that indicates which line the coordinate function is tied to. We could write f_L for a coordinate function on line L. With that notation, the symbol $f_L(P)$ would denote the coordinate of point P on line L. But although it might be clearer, we do not use the symbol f_L. We just use the symbol f.

Definition 22: Distance Function on the set of Real Numbers (page 72)
Words: The Distance Function on the Set of Real Numbers
Meaning: The function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x,y) = |x - y|$.

Definition 23: betweenness for real numbers (page 97)
words: "y is between x and z", where x, y, and z are real numbers.
symbol: $x * y * z$, where x, y, and z are real numbers
meaning: $x < y < z$ or $z < y < x$.
additional symbol: the symbol $w * x * y * z$ means $w < x < y < z$ or $z < y < x < w$, etc.

Definition 24: betweenness of points (page 100)
words: "Q is between P and R", where P, Q, R are points.
symbol: $P * Q * R$, where P, Q, R are points.
meaning: Points P, Q, R are collinear, lying on some line L, and there is a coordinate function f for line L such that the real number coordinate for Q is between the real number coordinates of P and R. That is, $f(P) * f(Q) * f(R)$.
remark: By Theorem 14, we know that it does not matter which coordinate function is used on line L. The betweenness property of the coordinates of the three points will be the same regardless of the coordinate function used.
additional symbol: The symbol $P * Q * R * S$ means $f(P) * f(Q) * f(R) * f(S)$, etc.

Definition 25: segment, ray (page 102)

words and symbols: "segment A, B", denoted $\overline{AB}$, and "ray A, B", denoted $\overrightarrow{AB}$

usage: A and B are distinct points.

meaning: Let f be a coordinate function for line $\overleftrightarrow{AB}$ with the property that $f(A) = 0$ and $f(B)$ is positive. (The existence of such a coordinate function is guaranteed by Theorem 11 (Ruler Placement Theorem).)

- Segment $\overline{AB}$ is the set $\overline{AB} = \{P \in \overleftrightarrow{AB} \; such \; that \; 0 \leq f(P) \leq f(B)\}$.
- Ray $\overrightarrow{AB}$ is the set $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} \; such \; that \; 0 \leq f(P)\}$.

additional terminology:

- Points A and B are called the *endpoints* of segment $\overline{AB}$.
- Point A is called the *endpoint* of ray $\overrightarrow{AB}$.
- The *length* of a segment is defined to be the distance between the endpoints. That is, $length(\overline{AB}) = d(A, B)$. As mentioned in Definition 20, many books use the symbol AB to denote $d(A, B)$. Thus we have the following choice of notations:

$$length(\overline{AB}) = d(A, B) = AB$$

Definition 26: Opposite rays are rays of the form $\overrightarrow{BA}$ and $\overrightarrow{BC}$ where $A * B * C$. (page 103)

Definition 27: angle (page 103)

words: "angle A, B, C"

symbol: $\angle ABC$

usage: A, B, C are non-collinear points.

meaning: Angle A, B, C is defined to be the following set: $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$

additional terminology: Point B is called the *vertex* of the angle. Rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are each called a *side* of the angle.

Definition 28: triangle (page 104)

words: "triangle A, B, C"

symbol: ΔABC

usage: A, B, C are non-collinear points.

meaning: Triangle A, B, C is defined to be the following set: $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

additional terminology: Points A, B, C are each called a *vertex* of the triangle. Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CA}$ are each called a *side* of the triangle.

Definition 29: segment congruence (page 104)

Two line segments are said to be congruent if they have the same length. The symbol $\cong$ is used to indicate this. For example $\overline{AB} \cong \overline{CD}$ means $length(\overline{AB}) = length(\overline{CD})$. Of course, this can also be denoted $d(A, B) = d(C, D)$ or $AB = CD$.

Definition 30: reflexive property (page 105)

words: Relation $\mathcal{R}$ is *reflexive*.

usage: $\mathcal{R}$ is a relation on some set A.

meaning: Element of set A is related to itself.

abbreviated version: For every $x \in A$, the sentence ${}_x\mathcal{R}_x$ is true.

More concise abbreviaton: $\forall x \in A, \; {}_x\mathcal{R}_x$

Definition 31: symmetric property (page 105)
words: Relation $\mathcal{R}$ is *symmetric*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: If x is related to y, then y is related to x.
abbreviated version: For every $x, y \in A$, if ${}_x\mathcal{R}_y$ is true then ${}_y\mathcal{R}_x$ is also true.
More concise abbreviaton: $\forall x, y \in A, if\ \ {}_x\mathcal{R}_y\ then\ \ {}_y\mathcal{R}_x$

Definition 32: transitive property (page 105)
words: Relation $\mathcal{R}$ is *transitive*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: If x is related to y and y is related to z, then x is related to z.
abbreviated: For every $x, y, z \in A$, if ${}_x\mathcal{R}_y$ is true and ${}_y\mathcal{R}_z$ is true, then ${}_x\mathcal{R}_z$ is also true.
More concise abbreviaton: $\forall x, y \in A, if\ \ {}_x\mathcal{R}_y\ and\ \ {}_y\mathcal{R}_z\ then\ \ {}_x\mathcal{R}_z$

Definition 33: equivalence relation (page 105)
words: Relation $\mathcal{R}$ is an *equivalence relation*.
usage: $\mathcal{R}$ is a relation on some set A.
meaning: $\mathcal{R}$ is reflexive and symmetric and transitive.

Definition 34: midpoint of a segment (page 110)
Words: M is a midpoint of Segment A, B.
Meaning: M lies on $\overleftrightarrow{AB}$ and $MA = MB$.

Definition 35: convex set (page 118)
- Without names: A set is said to be *convex* if for any two distinct points that are elements of the set, the segment that has those two points as endpoints is a subset of the set.
- With names: Set S is said to be convex if for any two distinct points $P, Q \in S$, the segment $\overline{PQ} \subset S$.

Definition 36: same side, opposite side, edge of a half-plane. (page 119)
Two points are said to lie on the *same side* of a given line if they are both elements of the same half-plane created by that line. The two points are said to lie on *opposite sides* of the line if one point is an element of one half-plane and the other point is an element of the other. The line itself is called the *edge of the half-plane*.

Definition 37: Angle Interior (page 123)
Words: The interior of $\angle ABC$.
Symbol: $Interior(\angle ABC)$
Meaning: The set of all points D that satisfy both of the following conditions.
Points D and A are on the same side of line $\overleftrightarrow{BC}$.
Points D and C are on the same side of line $\overleftrightarrow{AB}$.
Meaning abbreviated in symbols: $Interior(\angle ABC) = H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A)$
Related term: The *exterior* of $\angle ABC$ is defined to be the set of points that do not lie on the angle or in its interior.

Definition 38: Triangle Interior (page 123)

Words: The interior of ΔABC.

Symbol: $Interior(\Delta ABC)$

Meaning: The set of all points D that satisfy all three of the following conditions.

Points D and A are on the same side of line $\overleftrightarrow{BC}$.
Points D and B are on the same side of line $\overleftrightarrow{CA}$.
Points D and C are on the same side of line $\overleftrightarrow{AB}$.

Meaning abbreviated in symbols: $Interior(\Delta ABC) = H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A) \cap H_{\overleftrightarrow{CA}}(B)$.

Related term: The exterior of ΔABC is defined to be the set of points that do not lie on the triangle or in its interior.

Definition 39: quadrilateral (page 129)

words: "quadrilateral A, B, C, D"

symbol: $\square ABCD$

usage: A, B, C, D are distinct points, no three of which are collinear, and such that the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ intersect only at their endpoints.

meaning: quadrilateral A, B, C, D is the set $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$

additional terminology: Points A, B, C, D are each called a *vertex* of the quadrilateral.
Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CD}$ and $\overline{DA}$ are each called a *side* of the quadrilateral.
Segments $\overline{AC}$ and $\overline{BD}$ are each called a *diagonal* of the quadrilateral.

Definition 40: convex quadrilateral (page 132)

A *convex* quadrilateral is one in which all the points of any given side lie on the same side of the line determined by the opposite side. A quadrilateral that does not have this property is called *non-convex*.

Definition 41: The set of all abstract angles is denoted by the symbol $\mathcal{A}$. (page 139)

Definition 42: adjacent angles (page 141)

Two angles are said to be *adjacent* if they share a side but have disjoint interiors. That is, the two angles can be written in the form $\angle ABD$ and $\angle DBC$, where point C is not in the interior of $\angle ABD$ and point A is not in the interior of $\angle DBC$.

Definition 43: angle bisector (page 142)

An angle bisector is a ray that has its endpoint at the vertex of the angle and passes through a point in the interior of the angle, such that the two adjacent angles created have equal measure. That is, for an angle $\angle ABC$, a bisector is a ray $\overrightarrow{BD}$ such that $D \in interior(\angle ABC)$ and such that $m(\angle ABD) = m(\angle DBC)$.

Definition 44: linear pair (page 144)

Two angles are said to be a *linear pair* if they share one side, and the sides that they do not share are opposite rays. That is, if the two angles can be written in the form $\angle ABD$ and $\angle DBC$, where $A * B * C$.

Definition 45: vertical pair (page 146)

A vertical pair is a pair of angles with the property that the sides of one angle are the opposite rays of the sides of the other angle.

Definition 46: acute angle, right angle, obtuse angle (page 149)
An acute angle is an angle with measure less than 90.
A right angle is an angle with measure 90.
An obtuse angle is an angle with measure greater than 90.

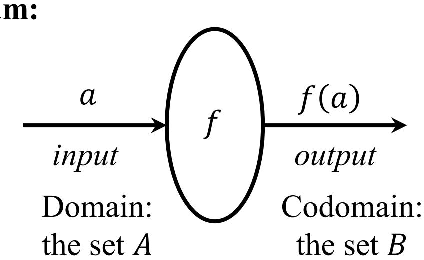
Definition 47: perpendicular lines (page 149)
Two lines are said to be perpendicular if there exist two rays that lie in the lines and whose union is a right angle. The symbol $L \perp M$ is used to denote that lines L and M are perpendicular.

Definition 48: perpendicular lines, segments, rays (page 149)
Suppose that Object 1 is a line or a segment or a ray and that Object 2 is a line or a segment or a ray. Object 1 is said to be perpendicular to Object 2 if the line that contains Object 1 is perpendicular to the line that contains Object 2 by the definition of perpendicular lines in the previous definition. The symbol $L \perp M$ is used to denote that objects L and M are perpendicular.

Definition 49: angle congruence (page 152)
Two angles are said to be *congruent* if they have the same measure. The symbol $\cong$ is used to indicate this. For example $\angle ABC \cong \angle DEF$ means $m(\angle ABC) = m(\angle DEF)$.

Definition 50: symbol for *equality* of two sets (found on page 157)
Words: $S\ equals\ T$.
Symbol: $S = T$.
Usage: S and T are sets.
Meaning: S and T are the same set. That is, every element of set S is also an element of set T, and vice-versa.

Definition 51: function, domain, codomain, image, machine diagram, correspondence (found on page 157)
Symbol: $f: A \to B$
Spoken: "f is a function that maps A to B."
Usage: A and B are sets. Set A is called the *domain* and set B is called the *codomain*.
Meaning: f is a machine that takes an element of set A as input and produces an element of set B as output.
More notation: If an element $a \in A$ is used as the input to the function , then the symbol $f(a)$ is used to denote the corresponding output. The output $f(a)$ is called *the image of* a *under the map* f.
Machine Diagram:



Additional notation: If f is both one-to-one and onto (that is, if f is a bijection), then the symbol $f: A \leftrightarrow B$ will be used. In this case, f is called a *correspondence* between the sets A and B.

Definition 52: Correspondence between vertices of two triangles (found on page 159)
Words: "f is a correspondence between the vertices of triangles ΔABC and ΔDEF."
Meaning: f is a one-to-one, onto function with domain $\{A, B, C\}$ and codomain $\{D, E, F\}$.

Definition 53: corresponding parts of two triangles (found on page 160)
Words: Corresponding parts of triangles ΔABC and ΔDEF.
Usage: A correspondence between the vertices of triangles ΔABC and ΔDEF has been given.
Meaning: As discussed above, if a correspondence between the vertices of triangles ΔABC and ΔDEF has been given, then there is an automatic correspondence between the sides of triangle ΔABC and and the sides of triangle ΔDEF, and also between the angles of triangle ΔABC and the angles of ΔDEF, For example, if the correspondence between vertices were $(A, B, C) \leftrightarrow (D, E, F)$, then corresponding parts would be pairs such as the pair of sides $\overline{AB} \leftrightarrow \overline{DE}$ and the pair of angles $\angle ABC \leftrightarrow \angle DEF$.

Definition 54: triangle congruence (found on page 160)
To say that two triangles are *congruent* means that there exists a correspondence between the vertices of the two triangles such that corresponding parts of the two triangles are congruent. If a correspondence between vertices of two triangles has the property that corresponding parts are congruent, then the correspondence is called *a congruence.* That is, the expression *a congruence* refers to a particular correspondence of vertices that has the special property that corresponding parts of the triangles are congruent.

Definition 55: symbol for *a congruence* of two triangles (found on page 160)
Symbol: $\Delta ABC \cong \Delta DEF$.
Meaning: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ of vertices is a congruence.

Definition 56: scalene, isosceles, equilateral, equiangular triangles (found on page 164)
A *scalene triangle* is one in which no two sides are congruent.
An *isosceles triangle* is one in which at least two sides are congruent.
An *equilateral triangle* is one in which all three sides are congruent.
An *equiangular triangle* is one in which all three angles are congruent.

Definition 57: *exterior angle*, *remote interior angle* (found on page 170)
An *exterior angle* of a triangle is an angle that forms a linear pair with one of the angles of the triangle. Each of the two other angles of the triangle is called a *remote interior angle* for that exterior angle. For example, a triangle ΔABC has six exterior angles. One of these is $\angle CBD$, where D is a point such that $A * B * D$. For the exterior angle $\angle CBD$, the two remote interior angles are $\angle ACB$ and $\angle CAB$.

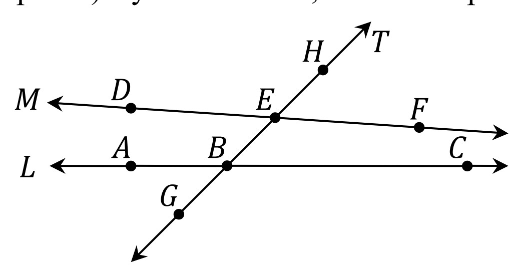
Definition 58: right triangle, and hypotenuse and legs of a right triangle (found on page 180)
A *right triangle* is one in which one of the angles is a right angle. Recall that Theorem 60 states that if a triangle has one right angle, then the other two angles are acute, so there can only be one right angle in a right triangle. In a right triangle, the side opposite the right angle

is called the *hypotenuse* of the triangle. Each of the other two sides is called a *leg* of the triangle.

Definition 59: altitude line, foot of an altitude line, altitude segment (found on page 181)
An *altitude line* of a triangle is a line that passes through a vertex of the triangle and is perpendicular to the opposite side. (Note that the altitude line does not necessarily have to intersect the opposite side to be perpendicular to it. Also note that Theorem 66 in the previous chapter tells us that there is exactly one altitude line for each vertex.) The point of intersection of the altitude line and the line determined by the opposite side is called the *foot of the altitude line*. An *altitude segment* has one endpoint at the vertex and the other endpoint at the foot of the altitude line drawn from that vertex. For example, in triangle ΔABC, an *altitude line* from vertex A is a line L that passes through A and is perpendicular to line $\overleftrightarrow{BC}$. The *foot* of altitude line L is the point D that is the intersection of line L and line $\overleftrightarrow{BC}$. The *altitude segment* from vertex A is the segment $\overline{AD}$. Point D can also be called the foot of the altitude segment $\overline{AD}$.

Definition 60: transversal (found on page 185)
Words: Line T is transversal to lines L and M.
Meaning: Line T intersects L and M in distinct points.

Definition 61: alternate interior angles, corresponding angles, interior angles on the same side of the transversal (found on page 185)
Usage: Lines L, M, and transversal T are given.
Labeled Points:
Let B be the intersection of lines T and L, and let E be the intersection of lines T and M. (By definition of transversal, B and E are not the same point.) By Theorem 15, there exist points A and C on line L such that $A * B * C$, points D and F on line M such that $D * E * F$, and points G and H on line T such that $G * B * E$ and $B * E * H$. Without loss of generality, we may assume that points D and F are labeled such that it is point D that is on the same side of line T as point A. (See the figure at right.)



Meaning:
Special names are given to the following eight pairs of angles:

- The pair $\{\angle ABE, \angle FEB\}$ is a pair of alternate interior angles.
- The pair $\{\angle CBE, \angle DEB\}$ is a pair of alternate interior angles.
- The pair $\{\angle ABG, \angle DEG\}$ is a pair of corresponding angles.
- The pair $\{\angle ABH, \angle DEH\}$ is a pair of corresponding angles.
- The pair $\{\angle CBG, \angle FEG\}$ is a pair of corresponding angles.
- The pair $\{\angle CBH, \angle FEH\}$ is a pair of corresponding angles.
- The pair $\{\angle ABE, \angle DEB\}$ is a pair of interior angles on the same side of the transversal.
- The pair $\{\angle CBE, \angle FEB\}$ is a pair of interior angles on the same side of the transversal.

Definition 62: special angle property for two lines and a transversal (found on page 188)
Words: Lines L and M and transversal T have the special angle property.
Meaning: The eight statements listed in Theorem 73 are true. That is, each pair of alternate interior angles is congruent. And each pair of corresponding angles is congruent. And each pair of interior angles on the same side of the transversal has measures that add up to 180.

Definition 63: circle, center, radius, radial segment, interior, exterior (found on page 195)
Symbol: $Circle(P,r)$
Spoken: the circle centered at P with radius r
Usage: P is a point and r is a positive real number.
Meaning: The following set of points: $Circle(P,r) = \{Q \text{ such that } PQ = r\}$
Additional Terminology:
- The point P is called the *center* of the circle.
- The number r is called the *radius* of the circle.
- The *interior* is the set $Interior\big(Circle(P,r)\big) = \{Q \text{ such that } PQ < r\}$.
- The *exterior* is the set $Exterior\big(Circle(P,r)\big) = \{Q \text{ such that } PQ > r\}$.
- Two circles are said to be *congruent* if they have the same radius.
- Two circles are said to be *concentric* if they have the same center.

Definition 64: Tangent Line and Secant Line for a Circle (found on page 195)
A tangent line for a circle is a line that intersects the circle at exactly one point.
A secant line for a circle is a line that intersects the circle at exactly two points.

Definition 65: chord, diameter segment, diameter, radial segment (found on page 197)
- A *chord* of a circle is a line segment whose endpoints both lie on the circle.
- A *diameter segment* for a circle is a chord that passes through the center of the circle.
- The *diameter* of a circle is the number $d = 2r$. That is, the diameter is the number that is the length of a diameter segment.
- A *radial segment* for a circle is a segment that has one endpoint at the center of the circle and the other endpoint on the circle. (So that the radius is the number that is the length of a radial segment.)

Definition 66: median segment and median line of a triangle (found on page 199)
A *median line* for a triangle is a line that passes through a vertex and the midpoint of the opposite side. A *median segment* for triangle is a segment that has its endpoints at those points.

Definition 67: incenter of a triangle (found on page 203)
The incenter of a triangle in Neutral Geometry is defined to be the point where the three angle bisectors meet. (Such a point is guaranteed to exist by Theorem 92.)

Definition 68: inscribed circle (found on page 204)
An *inscribed circle* for a polygon is a circle that has the property that each of the sides of the polygon is tangent to the circle.

Definition 69: tangent circles (found on page 205)
Two circles are said to be tangent to each other if they intersect in exactly one point.

Definition 70: The Axiom System for Euclidean Geometry (found on page 209)

Primitive Objects: *point*, *line*

Primitive Relation: *the point lies on the line*

Axioms of Incidence and Distance

<N1> There exist two distinct points. (at least two)

<N2> For every pair of distinct points, there exists exactly one line that both points lie on.

<N3> For every line, there exists a point that does not lie on the line. (at least one)

<N4> (The Distance Axiom) There exists a function $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, called the Distance Function on the Set of Points.

<N5> (The Ruler Axiom) Every line has a coordinate function.

Axiom of Separation

<N6> (The Plane Separation Axiom) For every line L, there are two associated sets called half-planes, denoted H_1 and H_2, with the following three properties:

(i) The three sets L, H_1, H_2 form a partition of the set of all points.

(ii) Each of the half-planes is convex.

(iii) If point P is in H_1 and point Q is in H_2, then segment $\overline{PQ}$ intersects line L.

Axioms of Angle measurement

<N7> (Angle Measurement Axiom) There exists a function $m: \mathcal{A} \to (0,180)$, called the Angle Measurement Function.

<N8> (Angle Construction Axiom) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $m(\angle PAB) = r$.

<N9> (Angle Measure Addition Axiom) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Axiom of Triangle Congruence

<N10> (*SAS* Axiom) If there is a one-to-one correspondence between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Euclidean Parallel Axiom

<EPA> (*EPA* Axiom) For any line L and any point P not on L, there is *not more than one* line M that passes through P and is parallel to L.

Definition 71: circumcenter of a triangle (found on page 215)

The circumcenter of a triangle in Euclidean Geometry is defined to be the point where the perpendicular bisectors of the three sides intersect. (Such a point is guaranteed to exist by Theorem 106)

Definition 72: a circle circumscribes a triangle (found on page 215)

We say that a circle *circumscribes* a triangle if the circle passes through all three vertices of the triangle.

Definition 73: parallelogram (found on page 215)

A parallelogram is a quadrilateral with the property that both pairs of opposite sides are parallel.

Definition 74: midsegment of a triangle (found on page 217)
A midsegment of a triangle is a line segment that has endpoints at the midpoints of two of the sides of the triangle.

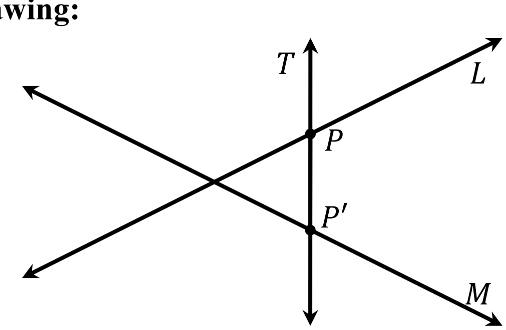
Definition 75: medial triangle (found on page 217)
Words: Triangle #1 is the medial triangle of triangle #2.
Meaning: The vertices of triangle #1 are the midpoints of the sides of triangle #2.
Additional Terminology: We will refer to triangle #2 as the outer triangle.

Definition 76: Orthocenter of a triangle in Euclidean Geometry (found on page 219)
The orthocenter of a triangle in Euclidean Geometry is a point where the three altitude lines intersect. (The existence of such a point is guaranteed by Theorem 113.)

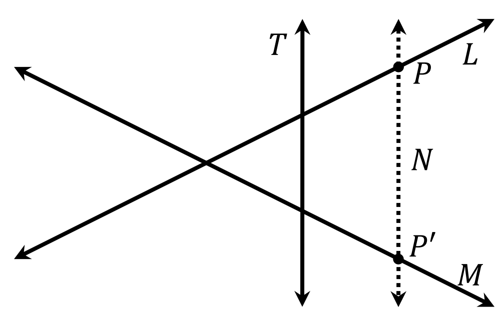
Definition 77: Equally-Spaced Parallel Lines in Euclidean Geometry (found on page 220)
Words: lines $L_1, L_2, \cdots, L_n$ are equally-spaced parallel lines.
Meaning: The lines are parallel and $L_1L_2 = L_2L_3 = \cdots = L_{n-1}L_n$.

Definition 78: Centroid of a triangle in Euclidean Geometry (found on page 222)
The centroid of a triangle in Euclidean Geometry is the point where the three medians intersect. (Such a point is guaranteed to exist by Theorem 116.)

Definition 79: Parallel Projection in Euclidean Geometry (found on page 225)
Symbol: $Proj_{L,M,T}$
Usage: L, M, T are lines, and T intersects both L and M.
Meaning: $Proj_{L,M,T}$ is a function whose domain is the set of points on line L and whose codomain is the set of points on line M. In function notation, this would be denoted by the symbol $Proj_{L,M,T}: L \to M$. Given an input point P on line L, the output point on line M is denoted P'. That is, $P' = Proj_{L,M,T}(P)$. The output point P' is determined in the following way:
Case 1: If P happens to lie at the intersection of lines L and T, then P' is defined to be the point at the intersection of lines M and T.
Case 2: If P lies on L but not on T, then there exists exactly one line N that passes through P and is parallel to line T. (Such a line N is guaranteed by Theorem 97). The output point P' is defined to be the point at the intersection of lines M and N.
Drawing:



Case 1: P lies on both L and T.



Case 2: P lies on L but not on T.

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Definition 80: triangle similarity (found on page 230)

To say that two triangles are *similar* means that there exists a correspondence between the vertices of the two triangles and the correspondence has these two properties:

- Each pair of corresponding angles is congruent.
- The ratios of the lengths of each pair of corresponding sides is the same.

If a correspondence between vertices of two triangles has the two properties, then the correspondence is called *a similarity*. That is, the expression *a similarity* refers to a particular correspondence of vertices that has the two properties.

Definition 81: symbol for *a similarity* of two triangles (found on page 231)

Symbol: $\Delta ABC \sim \Delta DEF$.

Meaning: The correspondence $(A, B, C) \leftrightarrow (D, E, F)$ of vertices is a similarity.

Definition 82: base times height (found on page 238)

For each side of a triangle, there is an opposite vertex, and there is an altitude segment drawn from that opposite vertex. The expression "$base\ times\ height$" or "$base \cdot height$" refers to the product of the length of a side of a triangle and the length of the corresponding altitude segment drawn to that side. The expression can be abbreviated $b \cdot h$.

Definition 83: triangular region, interior of a triangular region, boundary of a triangular region (found on page 243)

Symbol: $\blacktriangle ABC$

Spoken: triangular region A, B, C

Usage: A, B, C are non-collinear points

Meaning: the union of triangle ΔABC and the interior of triangle ΔABC. In symbols, we would write $\blacktriangle ABC = \Delta ABC \cup Interior(\Delta ABC)$.

Additional Terminology: the interior of a triangular region is defined to be the interior of the associated triangle. That is, $Interior(\blacktriangle ABC) = Interior(\Delta ABC)$. The boundary of a triangular region is defined to be the associated triangle, itself. That is, $Boundary(\blacktriangle ABC) = \Delta ABC$..

Definition 84: the set of all triangular regions is denoted by $\mathcal{R}$ **.** (found on page 244)

Definition 85: the area function for triangular regions (found on page 244)

symbol: $Area_{\blacktriangle}$

spoken: the area function for triangular regions

meaning: the function $Area_{\blacktriangle} : \mathcal{R}_{\blacktriangle} \to \mathbb{R}^{+}$ defined by $Area_{\blacktriangle}(\blacktriangle ABC) = \frac{bh}{2}$, where b is the length of any side of ΔABC and h is the length of the corresponding altitude segment. (Theorem 132 guarantees that the resulting value does not depend on the choice of base.)

Definition 86: polygon (found on page)

words: polygon $P_1, P_2, \ldots, P_n$

symbol: $Polygon(P_1 P_2 \ldots P_n)$

usage: $P_1, P_2, \ldots, P_n$ are distinct points, with no three in a row being collinear, and such that the segments $\overline{P_1 P_2}, \overline{P_1 P_2}, \ldots, \overline{P_n P_1}$ intersect only at their endpoints.

meaning: $Polygon(P_1 P_2 \ldots P_n)$ is defined to be the following set:

$$Polygon(P_1P_2 \dots P_n) = \overline{P_1P_2} \cup \overline{P_1P_2} \cup \dots \cup \overline{P_nP_1}$$

additional terminology: Points $P_1, P_2, \dots, P_n$ are each called a *vertex* of the polygon. Pairs of vertices of the form $\{P_k, P_{k+1}\}$ and the pair $\{P_n, P_1\}$ are called adjacent vertices. The n segments $\overline{P_1P_2}, \overline{P_1P_2}, \dots, \overline{P_nP_1}$ whose endpoints are adjacent vertices are each called a *side* of the polygon. Segments whose endpoints are non-adjacent vertices are each called a *diagonal* of the polygon.

Definition 87: convex polygon (found on page 247)

A *convex* polygon is one in which all the vertices that are not the endpoints of a given side lie in the same half-plane determined by that side. A polygon that does not have this property is called *non-convex*.

Definition 88: complex, polygonal region, separated, connected polygonal regions (found on page 247)

A *complex* is a finite set of triangular regions whose interiors do not intersect. That is, a set of the form $C = \{\blacktriangle_1, \blacktriangle_2, \dots, \blacktriangle_k\}$ where each $\blacktriangle_i$ is a triangular region and such that if $i \neq j$, then the intersection $Interior(\blacktriangle_i) \cap Interior(\blacktriangle_j)$ is the empty set.

A *polygonal region* is a set of points that can be described as the union of the triangular regions in a complex. That is a set of the form

$$R = \blacktriangle_1 \cup \blacktriangle_2 \cup \dots \cup \blacktriangle_k = \bigcup_{i=1}^{k} \blacktriangle_i$$

We say that a polygonal region can be *separated* if it can be written as the union of two disjoint polygonal regions. A *connected polygonal region* is one that cannot be separated into two disjoint polygonal regions. We will often use notation like $Region(P_1P_2 \dots P_n)$ to denote a connected polygonal region. In that symbol, the letters $P_1, P_2, \dots, P_n$ are vertices of the region (I won't give a precise definition of vertex. You get the idea.)

Definition 89: open disk, closed disk (found on page 248)

symbol: $disk(P,r)$

spoken: the open disk centered at point P with radius r.

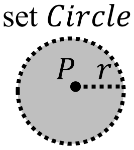
meaning: the set $Interior(Circle(P,r))$. That is, the set $\{Q: distance(P,Q) < r\}$.

another symbol: $\overline{disk(P,r)}$

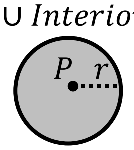
spoken: the closed disk centered at point P with radius r.

meaning: the set $Circle(P,r) \cup Interior(Circle(P,r))$. That is, $\{Q: distance(P,Q) \leq r\}$.

pictures:



the open disk $disk(P,r)$



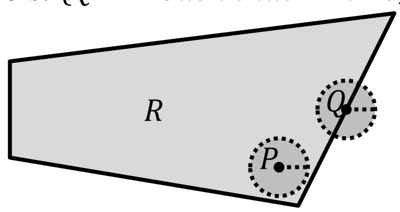
the closed disk $\overline{disk(P,r)}$

Definition 90: interior of a polygonal region, boundary of a polygonal region (page 249)

words: the interior of polygonal region R

meaning: the set of all points P in R with the property that there exists some open disk centered at point P that is entirely contained in R

meaning in symbols: $\{P \in R \text{ such that } \exists r > 0 \text{ such that } disk(P, r) \subset R\}$
additional terminology: the boundary of polygonal region R
meaning: the set of all points Q in R with the property that no open disk centered at point Q is entirely contained in R. This implies that every open disk centered at point Q contains some points that are not elements of the region R.
meaning in symbols: $\{Q \in R \text{ such that } \forall r > 0, disk(P, r) \not\subset R\}$
picture:



P is an interior point; Q is a boundary point

Definition 91: the set of all polygonal regions is denoted by $\mathcal{R}$. (found on page 250)

Definition 92: the area function for polygonal regions (found on page 251)
spoken: the area function for polygonal regions
meaning: the function $Area: \mathcal{R} \to \mathbb{R}^+$ defined by

$$Area(R) = Area_{\blacktriangle}(\blacktriangle_1) + Area_{\blacktriangle}(\blacktriangle_2) + \cdots + Area_{\blacktriangle}(\blacktriangle_3) = \sum_{i=1}^{k} Area_{\blacktriangle}(\blacktriangle_i)$$

where $C = \{\blacktriangle_1, \blacktriangle_2, \dots, \blacktriangle_k\}$ is a complex for region R. (Theorem 133 guarantees that the resulting value does not depend on the choice of complex C.)

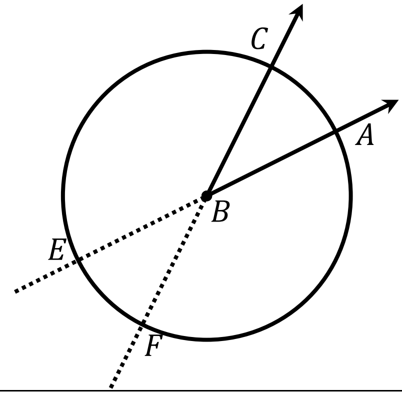
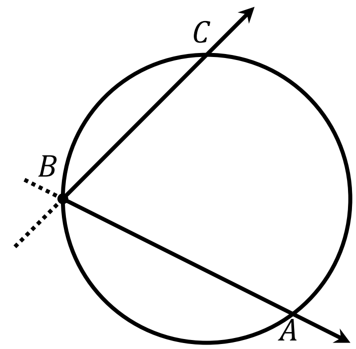
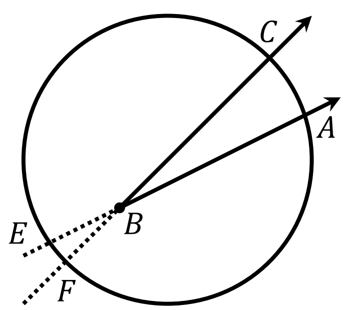
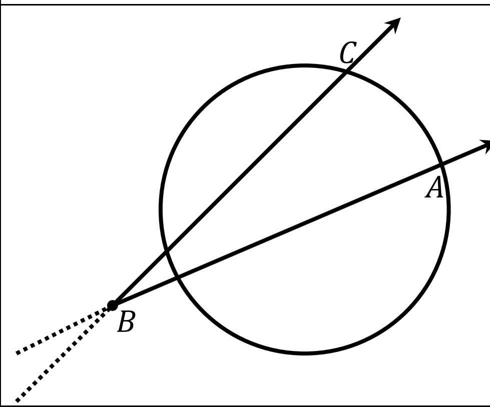
Definition 93: polygon similarity (found on page 254)
To say that two polygons are *similar* means that there exists a correspondence between the vertices of the two polygons and the correspondence has these two properties:

- Each pair of corresponding angles is congruent.
- The ratios of the lengths of each pair of corresponding sides is the same.

If a correspondence between vertices of two polygons has the two properties, then the correspondence is called *a similarity.* That is, the expression *a similarity* refers to a particular correspondence of vertices that has the two properties.

Definition 94: symbol for *a similarity* of two polygons (found on page 254)
Symbol: $Polygon(P_1P_2 \dots P_n) \sim Polygon(P_1'P_2' \dots P_n')$.
Meaning: The correspondence $(P_1P_2 \dots P_n) \leftrightarrow (P_1'P_2' \dots P_n')$ of vertices is a similarity.

Definition 95: seven types of angles intersecting circles (found on page 263)

Type 1 Angle (*Central Angle*) A *central angle* of a circle is an angle whose rays lie on two secant lines that intersect at the center of the circle. In the picture at right, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CF}$ are secant lines that intersect at the center point B of the circle. Angle $\angle ABC$ is a central angle. So are angles $\angle CBE, \angle EBF, \angle FBA$.	
Type 2 Angle (*Inscribed Angle*) An *inscribed angle* of a circle is an angle whose rays lie on two secant lines that intersect on the circle and such that each ray of the angle intersects the circle at one other point. In other words, an angle of the form $\angle ABC$, where A, B, C are three points on the circle. In the picture at right, angle $\angle ABC$ is an inscribed angle.	
Type 3 Angle Our third type of an angle intersecting a circle is an angle whose rays lie on two secant lines that intersect at a point that is inside the circle but is not the center of the circle. In the picture at right, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CF}$ are secant lines that intersect at point B in the interior of the circle. Angle $\angle ABC$ is an angle of type three. So are angles $\angle CBE, \angle EBF, \angle FBA$.	
Type 4 Angle Our fourth type of an angle intersecting a circle is an angle whose rays lie on two secant lines that intersect at a point that is outside the circle and such that each ray of the angle intersects the circle. In the picture at right, angle $\angle ABC$ is an angle of type four.	

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Type 5 Angle

Our fifth type of an angle intersecting a circle is an angle whose rays lie on two tangent lines and such that each ray of the angle intersects the circle. Because the rays lie in tangent lines, we know that each ray intersects the circle exactly once.

In the picture at right, angle $\angle ABC$ is an angle of type five.
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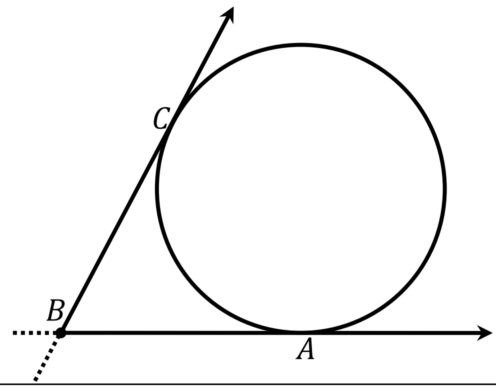

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Type 6 Angle

Our sixth type of an angle intersecting a circle is an angle whose vertex lies on the circle and such that one ray contains a chord of the circle and the other ray lies in a line that is tangent to the circle.

In the picture at right, angle $\angle ABC$ is an angle of type six.
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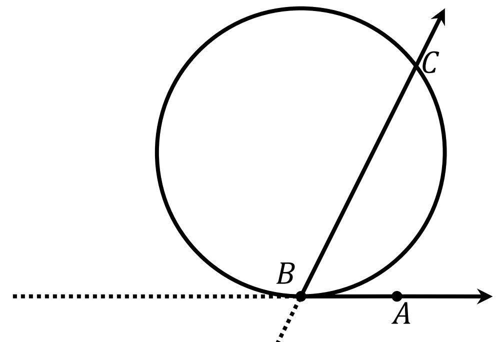

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Type 7 Angle

Our seventh type of an angle intersecting a circle is an angle whose rays lie on a secant line and tangent line that intersect outside the circle and such that each ray of the angle intersects the circle.

In the picture at right, angle $\angle ABC$ is an angle of type seven.
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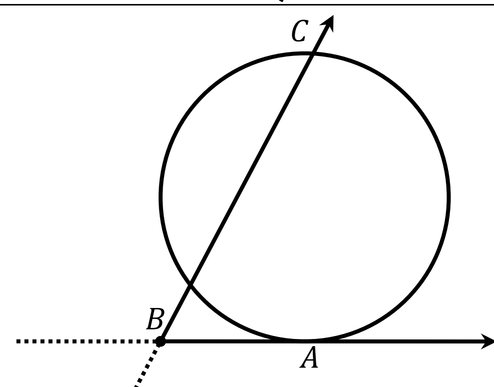

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Definition 96: Circular Arc (found on page 265)

Symbol: $\widehat{ABC}$

Spoken: arc A, B, C

Usage: A, B, C are non-collinear points.

Meaning: the set consisting of points A and C and all points of $Circle(A, B, C)$ that lie on the same side of line $\overleftrightarrow{AC}$ as point B.

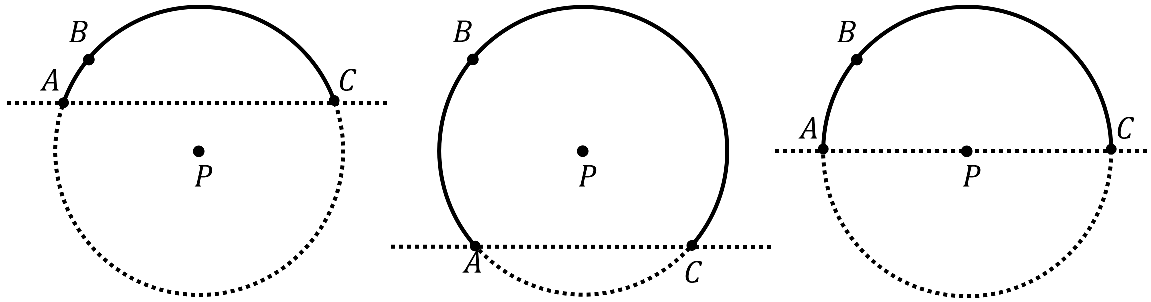
Meaning in Symbols: $\widehat{ABC} = \{A \cup C \cup (Circle(A, B, C) \cap H_B)\}$

Additional terminology:

- Points A and C are called the endpoints of arc $\widehat{ABC}$.
- The interior of the arc is the set $Circle(A, B, C) \cap H_B$.
- If the center P lies on the opposite side of line $\overleftrightarrow{AC}$ from point B, then arc $\widehat{ABC}$ is called a minor arc.
- If the center P of $Circle(A, B, C)$ lies on the same side of line $\overleftrightarrow{AC}$ as point B,then arc $\widehat{ABC}$ is called a major arc.

- If the center P lies on line $\overleftrightarrow{AC}$, then arc $\widehat{ABC}$ is called a semicircle.

Picture:



minor arc | major arc | semicircle

Definition 97: angle intercepting an arc (found on page 265)

We say that an angle intercepts an arc if each ray of the angle contains at least one endpoint of the arc and if the interior of the arc lies in the interior of the angle.

Definition 98: the symbol for the set of all circular arcs is $\hat{\mathcal{A}}$. (found on page 267)

Definition 99: the angle measure of an arc (found on page 267)

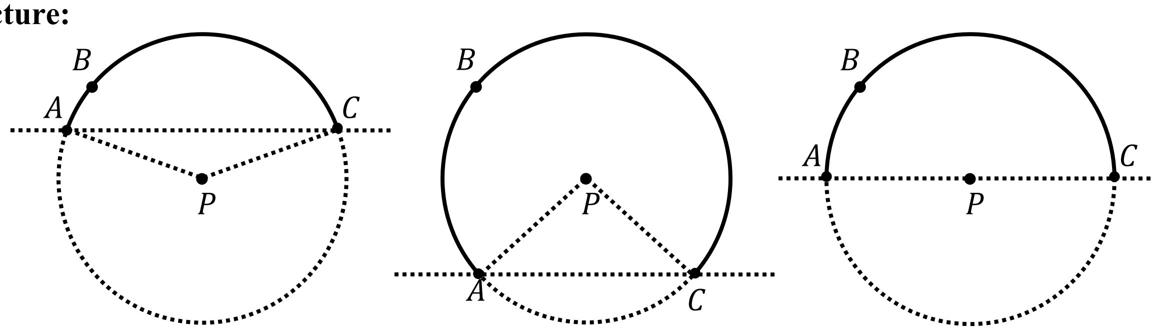
Symbol: $\widehat{m}$

Name: the Arc Angle Measurement Function

Meaning: The function $\widehat{m}: \hat{\mathcal{A}} \to (0,360)$, defined in the following way:

- If $\widehat{ABC}$ is a minor arc, then $\widehat{m}\left(\widehat{ABC}\right) = m(\angle APC)$, where point P is the center of the circle.
- If $\widehat{ABC}$ is a major arc, then $\widehat{m}\left(\widehat{ABC}\right) = 360 - m(\angle APC)$, where point P is the center of the circle.
- If $\widehat{ABC}$ is a semicircle, then $\widehat{m}\left(\widehat{ABC}\right) = 180$.

Picture:



minor arc $\widehat{m}\left(\widehat{ABC}\right) = m(\angle APC)$ | major arc $\widehat{m}\left(\widehat{ABC}\right) = 360 - m(\angle APC)$ | semicircle $\widehat{m}\left(\widehat{ABC}\right) = 180$

Definition 100: cyclic quadrilateral (found on page 277)

A quadrilateral is said to be cyclic if the quadrilateral can be circumscribed. That is, if there exists a circle that passes through all four vertices of the quadrilateral.

Definition 101: The Euler Line of a non-equilateral triangle (found on page 287)

Given a non-equilateral triangle, the Euler Line is defined to be the line containing the orthocenter, centroid, and circumcenter of that triangle. (Existence and uniqueness of this line s guaranteed by Theorem 157.)

Definition 102: The three Euler Points of a triangle are defined to be the midpoints of the segments connecting the vertices to the orthocenter. (found on page 287)

Definition 103: The *nine point circle* associated to a triangle is the circle that passes through the midpoints of the three sides, the feet of the three altitudes, and the three Euler points. (The existence of the nine point circle is guaranteed by Theorem 158.)

Definition 104: circumference of a circle and area of a circular region (found on page 291)

Given a circle of diameter d, for each $k = 1,2,3, \dots$ define $Poly_k$ to be a polygonal region bounded by a regular polygon with $3 \cdot 2^k$ sides, inscribed in the circle. Define P_k and A_k to be the perimeter and area of the k^{th} polygonal region. The resulting sequences $\{P_k\}$ and $\{A_k\}$ are increasing and bounded above and so they each have a limit.

- Define the circumference of the circle to be the real number $C = \lim_{k\to\infty} P_k$.
- Define the area of the circular region to be the real number $A = \lim_{k\to\infty} A_k$.

Definition 105: The symbol pi, or π, denotes the real number that is the ratio $\frac{circumference}{diameter}$ for any circle. That is, $\pi = \frac{C}{d}$. (That this ratio is the same for all circles is guaranteed by Theorem 159, found on page 293.) (found on page 294)

Definition 106: arc length (found on page 299)

The length of an arc $\widehat{ABC}$ on a circle of radius r is defined to be the number

$$\hat{L}(\widehat{ABC}) = \frac{\hat{m}(\widehat{ABC})\pi r}{180}$$

Definition 107: Rules for computing area (found on page 299)

(1) The area of a triangular region is equal to one-half the base times the height. It does not matter which side of the triangle is chosen as the base.
(2) The area of a polygonal region is equal to the sum of the areas of the triangles in a complex for the region.
(3) The area of a circular region is πr^2.
(4) More generally, the area of a circular sector bounded by arc $\widehat{ABC}$ is $\pi r^2 \cdot \frac{\hat{m}(\widehat{ABC})}{360}$.
(5) Congruence property: If two regions have congruent boundaries, then the area of the two regions is the same.
(6) Additivity property: If a region is the union of two smaller regions whose interiors do not intersect, then the area of the whole region is equal to the sum of the two smaller regions.

Definition 108: Image and Preimage of a single element

If $f: A \to B$ and $a \in A$ is used as input to the function f, then the corresponding output $f(a) \in B$ is called the image of a.

If $f: A \to B$ and $b \in B$, then the preimage of b, denoted $f^{-1}(b)$, is the set of all elements of A whose image is b. That is, $f^{-1}(b) = \{a \in A \text{ such that } f(a) = b\}$.

Definition 109: Image of a Set and Preimage of a Set

If $f: A \to B$ and $S \subset A$, then the image of S, denoted $f(S)$, is the set of all elements of B that are images of elements of S. That is,

$$f(S) = \{b \in B \text{ such that } b = f(a) \text{ for some } a \in S\}$$

If $f: A \to B$ and $T \subset B$, then the preimage of T, denoted $f^{-1}(T)$, is the set of all elements of A whose images are elements of T. That is,

$$f^{-1}(T) = \{a \in A \text{ such that } f(a) \in T\}$$

Definition 110: composition of fuctions, composite function

Symbol: $g \circ f$

Spoken: "g circle f",or "g after f", or "g composed with f", or "g after f"

Usage: $f: A \to B$ and $g: B \to C$

Meaning: the function $g \circ f: A \to C$ defined by $g \circ f(a) = g\big(f(a)\big)$.

Additional terminology: A function of the form $g \circ f$ is called a *composite function*.

Definition 111: One-to-One Function

Words: The function $f: A \to B$ is *one-to-one*.

Alternate Words: The function $f: A \to B$ is *injective*.

Meaning in Words: Different inputs always produce different outputs.

Meaning in Symbols: $\forall x_1, x_2, if\ x_1 \neq x_2\ then\ f(x_1) \neq f(x_2)$.

Contrapositive: If two outputs are the same, then the inputs must have been the same.

Contrapositive in Symbols: $\forall x_1, x_2, if\ f(x_1) = f(x_2), then\ x_1 = x_2$.

Definition 112: Onto Function

Words: The function $f: A \to B$ is *onto*.

Alternate Words: The function $f: A \to B$ is *surjective*.

Meaning in Words: For every element of the codomain, there exists an element of the domain that will produce that element of the codomain as output.

Meaning in Symbols: $\forall y \in B, \exists x \in A\ such\ that\ f(x) = y$.

Definition 113: Bijection, One-to-One Correspondence

Words: *"The function f is a bijection"*, or *"the function f is bijective"*.

Alternate Words: The function f is a *one-to-one correspondence*.

Meaning: The function f is both one-to-one and onto.

Definition 114: Inverse Functions, Inverse Relations

Words: Functions f and g are *inverses* of one another.

Usage: $f: A \to B$ and $g: B \to A$

Meaning: f and g satisfy the following two properties, called inverse relations:

$$\forall a \in A, g \circ f(a) = a$$
$$\forall b \in B, f \circ g(b) = b$$

Additional Symbols and Terminology: Another way of saying that functions f and g are inverses of one another is to say that g is the inverse of f. Instead of using different letters for a function and its inverse, it is common to use the symbol f^{-1} to denote the inverse of a function f. With this notation, we would say that $f: A \to B$ and $f^{-1}: B \to A$, and the inverse relations become:

$$\forall a \in A, f^{-1} \circ f(a) = a$$
$$\forall b \in B, f \circ f^{-1}(b) = b$$

Definition 115: The plane is defined to be the set $\mathcal{P}$ of all points.

Definition 116: A map of the plane is defined to be a function $f: \mathcal{P} \rightarrow \mathcal{P}$.

Definition 117: A transformation of the plane is defined to be a bijective map of the plane. The set of all transformations of the plane is denoted by the symbol T.

Definition 118: Isometry of the Plane

Words: f is an isometry of the plane.

Meaning: f is a distance preserving map of the plane. That is, for all points P and Q, the distance from P to Q is the same as the distance from $f(P)$ to $f(Q)$.

Meaning in symbols: $\forall P, Q \in \mathcal{P}, d(P, Q) = d\big(f(P), f(Q)\big)$.

Definition 119: The Identity Map of the Plane is the map $id: \mathcal{P} \rightarrow \mathcal{P}$ defined by $id(Q) = Q$ for every point Q.

Definition 120: a Fixed Point of a Map of the Plane

Words: Q is a fixed point of the map f.

Meaning: $f(Q) = Q$

Definition 121: The Dilation of the Plane

Symbol: $D_{C,k}$

Spoken: The dilation centered at C with scaling factor k

Usage: C is a point, called the *center* of the dilation, and k is a positive real number.

Meaning: The map $D_{C,k}: \mathcal{P} \rightarrow \mathcal{P}$ defined as follows

The point C is a fixed point of $D_{C,k}$. That is, $D_{C,k}(C) = C$.

When a point $Q \neq C$ is used as input to the map $D_{C,k}$, the output is the unique point $Q' = D_{C,k}(Q)$ that has these two properties:

- Point Q' lies on ray $\overrightarrow{CQ}$
- The distance $d(C, Q') = kd(C, Q)$

(The existence and uniqueness of such a point Q' is guaranteed by the Congruent Segment Construction Theorem, (Theorem 24).)

Definition 122: The Reflection of the Plane

Symbol: M_L

Spoken: The reflection in line L

Usage: L is a line, called the *line of reflection*

Meaning: The map $M_L: \mathcal{P} \rightarrow \mathcal{P}$ defined as follows

Every point on the line L is a fixed point of M_L. That is, if $P \in L$ then $M_L(P) = P$.

When a point Q not on line L is used as input to the map M_L, the output is the unique point $Q' = M_L(Q)$ such that line L is the perpendicular bisector of segment $\overline{QQ'}$.

(The existence and uniqueness of such a point Q' is can be proven using the axioms and theorems of Neutral Geometry. You are asked to provide details in an exercise.)

Definition 123: A binary operation on a set S is a function $*: S \times S \rightarrow S$.

Definition 124: associativity, associative binary operation

Words: "$*$ is associative" or "$*$ has the associativity property"
Usage: $*$ is a binary operation $*$ on some set S.
Meaning: $\forall a, b, c \in S, a * (b * c) = (a * b) * c$

Definition 125: identity element, binary operation with an identity element
Words: "$*$ has an identity element."
Usage: $*$ is a binary operation $*$ on some set S.
Meaning:.There is an element $\exists e \in S$ with the following property: $\forall a \in S, a * e = e * a = a$
Meaning in symbols: $\exists e \in S: \forall a \in S, a * e = e * a = a$
Additional Terminology: The element $\exists e \in S$ is called *the identity for operation* $*$.

Definition 126: binary operation with inverses
Words: "$*$ has inverses."
Usage: $*$ is a binary operation $*$ on some set S.
Meaning: For each element $a \in S$, there exists is an $a^{-1} \in S$ such that

$$a * a^{-1} = e \text{ and } a^{-1} * a = e.$$

Meaning in symbols: $\forall a \in S, \exists a^{-1} \in S: a * a^{-1} = a^{-1} * a = e$
Additional Terminology: The element $a^{-1} \in S$ is called the *inverse* of a.

Definition 127: commutativity, commutative binary operation
Words: "$*$ is commutative" or "$*$ has the commutative property"
Usage: $*$ is a binary operation $*$ on some set S.
Meaning: $\forall a, b, c \in S, a * b = b * a$

Definition 128: Group
A Group is a pair $(G,*)$ consisting of a set G and a binary operation $*$ on G that has the following three properties.
(1) Associativity (Definition 124)
(2) Existence of an Identity Element (Definition 125)
(3) Existence of an Inverse for each Element (Definition 126)

Definition 129: Commutative Group, Abelian Group
A commutative group (or abelian group) is a group $(G,*)$ that has the commutativity property (Definition 127).

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Appendix 2: List of Theorems

Theorem 1: In Neutral Geometry, if L and M are distinct lines that intersect, then they intersect in only one point. (page 62)

Theorem 2: In Neutral Geometry, there exist three non-collinear points. (page 62)

Theorem 3: In Neutral Geometry, there exist three lines that are not concurrent. (page 62)

Theorem 4: In Neutral Geometry, for every point P, there exists a line that does not pass through P. (page 62)

Theorem 5: In Neutral Geometry, for every point P, there exist at least two lines that pass through P. (page 62)

Theorem 6: In Neutral Geometry, given any points P and Q that are not known to be distinct, there exists at least one line that passes through P and Q. (page 63)

Theorem 7: about how many points are on lines in Neutral Geometry (page 70)

In Neutral Geometry, given any line L, the set of points that lie on L is an infinite set. More precisely, the set of points that lie on L can be put in one-to-one correspondence with the set of real numbers $\mathbb{R}$. (In the terminology of sets, we would say that the set of points on line L has the same cardinality as the set of real numbers $\mathbb{R}$.)

Theorem 8: The Distance Function on the Set of Points, the function d, is Positive Definite. (page 74)

For all points P and Q, $d(P,Q) \geq 0$, and $d(P,Q) = 0$ if and only if $P = Q$. That is, if and only if P and Q are actually the same point.

Theorem 9: The Distance Function on the Set of Points, the function d, is Symmetric. (page 74)

For all points P and Q, $d(P,Q) = d(Q,P)$.

Theorem 10: (Ruler Sliding and Ruler Flipping) Lemma about obtaining a new coordinate function from a given one (page 87)

Suppose that $f: L \to \mathbb{R}$ is a coordinate function for a line L.

(A) (Ruler Sliding) If c is a real number constant and g is the function $g: L \to \mathbb{R}$ defined by $g(P) = f(P) + c$, then g is also a coordinate function for line L.

(B) (Ruler Flipping) If g is the function $g: L \to \mathbb{R}$ defined by $g(P) = -f(P)$, then g is also a coordinate function for line L.

Theorem 11: Ruler Placement Theorem (page 89)

If A and B are distinct points on some line L, then there exists a coordinate function h for line L such that $h(A) = 0$ and $h(B)$ is positive.

Theorem 12: facts about betweenness for real numbers (page 98)

(A) If $x * y * z$ then $z * y * x$.

(B) If x, y, z are three distinct real numbers, then exactly one is between the other two.
(C) Any four distinct real numbers can be named in an order w, x, y, z so that $w * x * y * z$.
(D) If a and b are distinct real numbers, then
(D.1) There exists a real number c such that $a * c * b$.
(D.2) There exists a real number d such that $a * b * d$

Theorem 13: Betweenness of real numbers is related to the distances between them. (page 98)
Claim: For distinct real numbers x, y, z, the following are equivalent
(A) $x * y * z$
(B) $|x - z| = |x - y| + |y - z|$. That is, $d_{\mathbb{R}}(x, z) = d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$.

Theorem 14: Lemma about betweenness of coordinates of three points on a line (page 99)
If P, Q, R are three distinct points on a line L, and f is a coordinate function on line L, and the betweenness expression $f(P) * f(Q) * f(R)$ is true, then for any coordinate function g on line L, the expression $g(P) * g(Q) * g(R)$ will be true.

Theorem 15: Properties of Betweenness for Points (page 100)
(A) If $P * Q * R$ then $R * Q * P$.
(B) For any three distinct collinear points, exactly one is between the other two.
(C) Any four distinct collinear points can be named in an order P, Q, R, S such that $P * Q * R * S$.
(D) If P and R are distinct points, then
(D.1) There exists a point Q such that $P * Q * R$.
(D.2) There exists a point S such that $P * R * S$.

Theorem 16: Betweenness of points on a line is related to the distances between them. (page 101)
Claim: For distinct collinear points P, Q, R, the following are equivalent
(A) $P * Q * R$
(B) $d(P, R) = d(P, Q) + d(Q, R)$.

Theorem 17: Lemma about distances between three distinct, collinear points. (page 102)
If P, Q, R are distinct collinear points such that $P * Q * R$ is not true, then the inequality $d(P, R) < d(P, Q) + d(Q, R)$ is true.

Theorem 18: (Corollary) Segment $\overline{AB}$ is a subset of ray $\overrightarrow{AB}$. (page 103)

Theorem 19: about the use of different second points in the symbol for a ray. (page 103)
If $\overrightarrow{AB}$ and C is any point of $\overrightarrow{AB}$ that is not A, then $\overrightarrow{AB} = \overrightarrow{AC}$.

Theorem 20: Segment congruence is an equivalence relation. (page 106)

Theorem 21: About a point whose coordinate is the average of the coordinates of the endpoints. (page 111)
Given $\overline{AB}$, and Point C on line $\overleftrightarrow{AB}$, and any coordinate function f for line $\overleftrightarrow{AB}$, the following are equivalent:

(i) The coordinate of point C is the average of the coordinates of points A and B. That is, $f(C) = \frac{f(A)+f(B)}{2}$.
(ii) Point C is a midpoint of segment $\overline{AB}$. That is, $CA = CB$.

Theorem 22: Corollary of Theorem 21. (page 112)
Given $\overline{AB}$, and Point C on line $\overleftrightarrow{AB}$, and any coordinate functions f and g for line $\overleftrightarrow{AB}$, the following are equivalent:
(i) $f(C) = \frac{f(A)+f(B)}{2}$.
(ii) $g(C) = \frac{g(A)+g(B)}{2}$.

Theorem 23: Every segment has exactly one midpoint. (page 112)

Theorem 24: Congruent Segment Construction Theorem. (page 113)
Given a segment $\overline{AB}$ and a ray $\overrightarrow{CD}$, there exists exactly one point E on ray $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Theorem 25: Congruent Segment Addition Theorem. (page 113)
If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.

Theorem 26: Congruent Segment Subtraction Theorem. (page 113)
If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$ then $\overline{BC} \cong \overline{B'C'}$.

Theorem 27: Given any line, each of its half-planes contains at least three non-collinear points. (page 120)

Theorem 28: (Pasch's Theorem) about a line intersecting a side of a triangle between vertices (page 122)
If a line intersects the side of a triangle at a point between vertices, then the line also intersects the triangle at another point that lies on at least one of the other two sides.

Theorem 29: about a line intersecting two sides of a triangle between vertices (page 122)
If a line intersects two sides of a triangle at points that are not vertices, then the line cannot intersect the third side.

Theorem 30: about a ray with an endpoint on a line (page 124)
If a ray has its endpoint on a line but does not lie in the line,
then all points of the ray except the endpoint are on the same side of the line.

Theorem 31: (Corollary of Theorem 30) about a ray with its endpoint on an angle vertex (p. 125)
If a ray has its endpoint on an angle vertex and passes through a point in the angle interior, then every point of the ray except the endpoint lies in the angle interior.

Theorem 32: (Corollary of Theorem 30.) about a segment that has an endpoint on a line (p. 125)
If a segment that has an endpoint on a line but does not lie in the line,

then all points of the segment except that endpoint are on the same side of the line.

Theorem 33: (Corollary of Theorem 32.) Points on a side of a triangle are in the interior of the opposite angle. (page 125)
If a point lies on the side of a triangle and is not one of the endpoints of that side, then the point is in the interior of the opposite angle.

Theorem 34: The *Z* Lemma (page 126)
If points C and D lie on opposite sides of line $\overleftrightarrow{AB}$, then ray $\overrightarrow{AC}$ does not intersect ray $\overrightarrow{BD}$.

Theorem 35: The Crossbar Theorem (page 126)
If point D is in the interior of $\angle ABC$, then $\overrightarrow{BD}$ intersects $\overline{AC}$ at a point between A and C.

Theorem 36: about a ray with its endpoint in the interior of a triangle (page 128)
If the endpoint of a ray lies in the interior of a triangle, then the ray intersects the triangle exactly once.

Theorem 37: about a line passing through a point in the interior of a triangle (page 128)
If a line passes through a point in the interior of a triangle, then the line intersects the triangle exactly twice.

Theorem 38: Three equivalent statements about quadrilaterals (page 130)
For any quadrilateral, the following statements are equivalent:
(i) All the points of any given side lie on the same side of the line determined by the opposite side.
(ii) The diagonal segments intersect.
(iii) Each vertex is in the interior of the opposite angle.

Theorem 39: about points in the interior of angles (page 141)
Given: points C and D on the same side of line $\overleftrightarrow{AB}$.
Claim: The following are equivalent:
(I) D is in the interior of $\angle ABC$.
(II) $m(\angle ABD) < m(\angle ABC)$.

Theorem 40: Every angle has a unique bisector. (page 143)

Theorem 41: Linear Pair Theorem. (page 144)
If two angles form a linear pair, then the sum of their measures is 180.

Theorem 42: Converse of the Linear Pair Theorem (page 146)
If adjacent angles have measures whose sum is 180, then the angles form a linear pair. That is, if angles $\angle ABD$ and $\angle DBC$ are adjacent and $m(\angle ABD) + m(\angle DBC) = 180$, then $A * B * C$.

Theorem 43: Vertical Pair Theorem (page 146)
If two angles form a vertical pair then they have the same measure.

Theorem 44: about angles with measure 90 (page 148)

For any angle, the following two statements are equivalent.

(i) There exists another angle that forms a linear pair with the given angle and that has the same measure.

(ii) The given angle has measure 90.

Theorem 45: If two intersecting lines form a right angle, then they actually form four. (page 150)

Theorem 46: existence and uniqueness of a line that is perpendicular to a given line through a given point that *lies on* the given line (page 150)

For any given line, and any given point that *lies on* the given line, there is exactly one line that passes through the given point and is perpendicular to the given line.

Theorem 47: Angle congruence is an equivalence relation. (page 152)

Theorem 48: Congruent Angle Construction Theorem (page 152)

Let $\overrightarrow{AB}$ be a ray on the edge of a half-plane H. For any angle $\angle CDE$, there is exactly one ray $\overrightarrow{AP}$ with point P in H such that $\angle PAB \cong \angle CDE$.

Theorem 49: Congruent Angle Addition Theorem (page 152)

If point D lies in the interior of $\angle ABC$ and point D' lies in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle DBC \cong \angle D'B'C'$, then $\angle ABC \cong \angle A'B'C'$.

Theorem 50: Congruent Angle Subtraction Theorem (page 152)

If point D lies in the interior of $\angle ABC$ and point D' lies in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle ABC \cong \angle A'B'C'$, then $\angle DBC \cong \angle D'B'C'$.

Theorem 51: triangle congruence is an equivalence relation (page 160)

Theorem 52: the *CS* ➔ *CA* theorem for triangles (the Isosceles Triangle Theorem) (page 164)

In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.

Theorem 53: (Corollary) In Neutral Geometry, if a triangle is equilateral then it is equiangular. (page 165)

Theorem 54: the *ASA* Congruence Theorem for Neutral Geometry (page 165)

In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Theorem 55: the *CA* ➔ *CS* theorem for triangles in Neutral Geometry (page 167)

In Neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if *CA* then *CS*.

Theorem 56: (Corollary) In Neutral Geometry, if a triangle is equiangular then it is equilateral. (page 167)

Theorem 57: (Corollary) The *CACS* theorem for triangles in Neutral Geometry. (page 168)
In any triangle in Neutral Geometry, congruent angles are always opposite congruent sides. That is, *CA* ⇔ *CS*.

Theorem 58: the *SSS* congruence theorem for Neutral Geometry (page 168)
In Neutral Geometry, if there is a one-to-one correspondence between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Theorem 59: Neutral Exterior Angle Theorem (page 170)
In Neutral Geometry, the measure of any exterior angle is greater than the measure of either of its remote interior angles.

Theorem 60: (Corollary) If a triangle has a right angle, then the other two angles are acute. (page 172)

Theorem 61: the *BS* ➔ *BA* theorem for triangles in Neutral Geometry (page 172)
In Neutral Geometry, if one side of a triangle is longer than another side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side. That is, in a triangle, if *BS* then *BA*.

Theorem 62: the *BA* ➔ *BS* theorem for triangles in Neutral Geometry (page 172)
In Neutral Geometry, if the measure of one angle is greater than the measure of another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle. That is, in a triangle, if *BA* then *BS*.

Theorem 63: (Corollary) The *BABS* theorem for triangles in Neutral Geometry. (page 173)
In any triangle in Neutral Geometry, bigger angles are always opposite bigger sides. That is, *BA* ⇔ *BS*.

Theorem 64: The Triangle Inequality for Neutral Geometry (page 174)
In Neutral Geometry, the length of any side of any triangle is less than the sum of the lengths of the other two sides.
That is, for all non-collinear points A, B, C, the inequality $AC < AB + BC$ is true.

Theorem 65: The Distance Function Triangle Inequality for Neutral Geometry (page 175)
The function d satisfies the *Distance Function Triangle Inequality*.
That is, for all points P, Q, R, the inequality $d(P, R) \leq d(P, Q) + d(Q, R)$ is true.

Theorem 66: existence and uniqueness of a line that is perpendicular to a given line through a given point that *does not lie on* the given line (page 178)
For any given line and any given point that *does not lie on* the given line, there is exactly one line that passes through the given point and is perpendicular to the given line.

Theorem 67: The shortest segment connecting a point to a line is the perpendicular. (page 180)

Theorem 68: In any right triangle in Neutral Geometry, the hypotenuse is the longest side. (page 181)

Theorem 69: (Lemma) In any triangle in Neutral Geometry, the altitude to a longest side intersects the longest side at a point between the endpoints. (page 181)

Given: Neutral Geometry triangle ΔABC, with point D the foot of the altitude line drawn from vertex C to line $\overleftrightarrow{AB}$.

Claim: If $\overline{AB}$ is the longest side (that is, if $AB > BC$ and $AB > BC$), then $A * D * B$.

Theorem 70: the Angle-Angle-Side (*AAS*) Congruence Theorem for Neutral Geometry (p.183) In Neutral Geometry, if there is a correspondence between parts of two right triangles such that two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Theorem 71: the Hypotenuse Leg Congruence Theorem for Neutral Geometry (page 183) In Neutral Geometry, if there is a one-to-one correspondence between the vertices of any two right triangles, and the hypotenuse and a side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Theorem 72: the Hinge Theorem for Neutral Geometry (page 185) In Neutral Geometry, if triangles ΔABC and ΔDEF have $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ and $m(\angle A) > m(\angle D)$, then $BC > EF$.

Theorem 73: Equivalent statements about angles formed by two lines and a transversal in Neutral Geometry (page 186)

Given: Neutral Geometry, lines L and M and a transversal T, with points $A, \cdots, H$ labeled as in Definition 61, above.

Claim: The following statements are equivalent:

(1) The first pair of alternate interior angles is congruent. That is, $\angle ABE \cong \angle FEB$.
(2) The second pair of alternate interior angles is congruent. That is, $\angle CBE \cong \angle DEB$.
(3) The first pair of corresponding angles is congruent. That is, $\angle ABG \cong \angle DEG$.
(4) The second pair of corresponding angles is congruent. That is, $\angle ABH \cong \angle DEH$.
(5) The third pair of corresponding angles is congruent. That is, $\angle CBG \cong \angle FEG$.
(6) The fourth pair of corresponding angles is congruent. That is, $\angle CBH \cong \angle FEH$.
(7) The first pair of interior angles on the same side of the transversal has measures that add up to 180. That is, $m(\angle ABE) + m(\angle DEB) = 180$.
(8) The second pair of interior angles on the same side of the transversal has measures that add up to 180. That is, $m(\angle CBE) + m(\angle FEB) = 180$.

Theorem 74: The Alternate Interior Angle Theorem for Neutral Geometry (page 188)

Given: Neutral Geometry, lines L and M and a transversal T

Claim: If a pair of alternate interior angles is congruent, then lines L and M are parallel.

Contrapositive: If L and M are not parallel, then a pair of alternate interior angles are not congruent.

Theorem 75: Corollary of The Alternate Interior Angle Theorem for Neutral Geometry (p. 189)
Given: Neutral Geometry, lines L and M and a transversal T
Claim: If any of the statements of Theorem 73 are true (that is, if lines L, M, T have the special angle property), then L and M are parallel .
Contrapositive: If L and M are not parallel, then all of the statements of Theorem 73 are false (that is, lines L, M, T do not have the special angle property).

Theorem 76: Existence of a parallel through a point P not on a line L in Neutral Geometry. (page 189)
In Neutral Geometry, for any line L and any point P not on L, there exists *at least one* line M that passes through P and is parallel to L.

Theorem 77: In Neutral Geometry, the Number of Possible Intersection Points for a Line and a Circle is 0, 1, 2. (page 195)

Theorem 78: In Neutral Geometry, tangent lines are perpendicular to the radial segment. (p.196)
Given: A segment $\overline{AB}$ and a line L passing through point B.
Claim: The following statements are equivalent.
(i) Line L is perpendicular to segment $\overline{AB}$.
(ii) Line L is tangent to $Circle(A, AB)$ at point B. That is, L only intersects $Circle(A, AB)$ at point B.

Theorem 79: (Corollary of Theorem 78) For any line tangent to a circle in Neutral Geometry, all points on the line except for the point of tangency lie in the circle's exterior. (page 197)

Theorem 80: about points on a Secant line lying in the interior or exterior in Neutral Geometry (page 197)
Given: $Circle(A, r)$ and a secant line L passing through points B and C on the circle
Claim:
(i) If $B * D * C$, then D is in the interior of the circle.
(ii) If $D * B * C$ or $B * C * D$, then D is in the exterior of the circle.

Theorem 81: (Corollary of Theorem 80) about points on a chord or radial segment that lie in the interior in Neutral Geometry (page 198)
In Neutral Geometry, all points of a segment except the endpoints lie in the interior of the circle. Furthermore, one endpoint of a radial segment lies on the circle; all the other points of a radial segment lie in the interior of the circle.

Theorem 82: In Neutral Geometry, if a line passes through a point in the interior of a circle, then it also passes through a point in the exterior. (page 198)

Theorem 83: In Neutral Geometry, if a line passes through a point in the interior of a circle and also through a point in the exterior, then it intersects the circle at a point between those two points. (page 198)

Theorem 84: (Corollary) In Neutral Geometry, if a line passes through a point in the interior of a circle, then the line must be a secant line. That is, the line must intersect the circle exactly twice. (page 198)

Theorem 85: about special rays in isosceles triangles in Neutral Geometry (page 199)
Given: Neutral Geometry, triangle ΔABC with $\overline{AB} \cong \overline{AC}$ and ray $\overrightarrow{AD}$ such that $B * D * C$.
Claim: The following three statements are equivalent.
(i) Ray $\overrightarrow{AD}$ is the bisector of angle $\angle BAC$.
(ii) Ray $\overrightarrow{AD}$ is perpendicular to side $\overline{BC}$.
(iii) Ray $\overrightarrow{AD}$ bisects side $\overline{BC}$. That is, point D is the midpoint of side $\overline{BC}$.

Theorem 86: about points equidistant from the endpoints of a line segment in Neutral Geometry (page 200)
In Neutral Geometry, the following two statements are equivalent
(i) A point is equidistant from the endpoints of a line segment.
(ii) The point lies on the perpendicular bisector of the segment.

Theorem 87: In Neutral Geometry, any perpendicular from the center of a circle to a chord bisects the chord (page 200)

Theorem 88: In Neutral Geometry, the segment joining the center to the midpoint of a chord is perpendicular to the chord. (page 200)

Theorem 89: (Corollary of Theorem 86) In Neutral Geometry, the perpendicular bisector of a chord passes through the center of the circle. (page 200)

Theorem 90: about chords equidistant from the centers of circles in Neutral Geometry (p. 201)
Given: Neutral Geometry, chord $\overline{BC}$ in $Circle(A, r)$ and chord $\overline{QR}$ in $Circle(P, r)$ with the same radius r.
Claim: The following two statements are equivalent:
(i) The distance from chord $\overline{BC}$ to center A is the same as the distance from chord $\overline{QR}$ to center P.
(ii) The chords have the same length. That is, $\overline{BC} \cong \overline{QR}$.

Theorem 91: about points on the bisector of an angle in Neutral Geometry (page 202)
Given: Neutral Geometry, angle $\angle BAC$, and point D in the interior of the angle
Claim: The following statements are equivalent
(i) D lies on the bisector of angle $\angle BAC$.
(ii) D is equidistant from the sides of angle $\angle BAC$.

Theorem 92: in Neutral Geometry, the three angle bisectors of any triangle are concurrent at a point that is equidistant from the three sides of the triangle. (page 202)

Theorem 93: about tangent lines drawn from an exterior point to a circle in Neutral Geometry (page 203)

Given: Neutral Geometry, $Circle(P,r)$, point A in the exterior of the circle, and lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ tangent to the circle at points B and C.
Claim: $\overline{AB} \cong \overline{AC}$ and $\angle PAB \cong \angle PAC$.

Theorem 94: In Neutral Geometry, if three points lie on a circle, then they do not lie on any other circle. (page 204)

Theorem 95: in Neutral Geometry, every triangle has exactly one inscribed circle. (page 204)

Theorem 96: In Neutral Geometry, if one circle passes through a point that is in the interior of another circle and also passes through a point that is in the exterior of the other circle, then the two circles intersect at exactly two points. (page 205)

Theorem 97: (Corollary) In Euclidean Geometry, the answer to the recurring question is *exactly one line*. (page 210)
In Euclidean Geometry, for any line L and any point P not on L, there exists *exactly one* line M that passes through P and is parallel to L.

Theorem 98: (corollary) In Euclidean Geometry, if a line intersects one of two parallel lines, then it also intersects the other. (page 210)
In Euclidean Geometry, if L and M are parallel lines, and line T intersects M, then T also intersects L.

Theorem 99: (corollary) In Euclidean Geometry, if two distinct lines are both parallel to a third line, then the two lines are parallel to each other. (page 211)
In Euclidean Geometry, if distinct lines M and N are both are parallel to line L, then M and N are parallel to each other.

Theorem 100: Converse of the Alternate Interior Angle Theorem for Euclidean Geometry (page 211)
Given: Euclidean Geometry, lines L and M and a transversal T
Claim: If L and M are parallel, then a pair of alternate interior angles is congruent

Theorem 101: (corollary) Converse of Theorem 75. (page 212)
Given: Euclidean Geometry, lines L and M and a transversal T
Claim: If L and M are parallel, then all of the statements of Theorem 73 are true (that is, lines L, M, T have the special angle property).

Theorem 102: (corollary) In Euclidean Geometry, if a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other. That is, if lines L and M are parallel, and line T is perpendicular to M, then T is also perpendicular to L. (page 212)

Theorem 103: In Euclidean Geometry, the angle sum for any triangle is 180. (page 212)

Theorem 104: (corollary) Euclidean Exterior Angle Theorem. (page 213)

In Euclidean Geometry, the measure of any exterior angle is equal to the sum of the measure of its remote interior angles

Theorem 105: (corollary) In Euclidean Geometry, the angle sum of any convex quadrilateral is 360. (page 213)

Theorem 106: In Euclidean Geometry, the perpendicular bisectors of the three sides of any triangle are concurrent at a point that is equidistant from the vertices of the triangle. (This point will be called the circumcenter.) (page 214)

Theorem 107: (corollary) In Euclidean Geometry, every triangle can be circumscribed. (p. 215)

Theorem 108: equivalent statements about convex quadrilaterals in Euclidean Geometry (p.216)
In Euclidean Geometry, given any convex quadrilateral, the following statements are equivalent (TFAE)
(i) Both pairs of opposite sides are parallel. That is, the quadrilateral is a parallelogram.
(ii) Both pairs of opposite sides are congruent.
(iii) One pair of opposite sides is both congruent and parallel.
(iv) Each pair of opposite angles is congruent.
(v) Either diagonal creates two congruent triangles.
(vi) The diagonals bisect each other.

Theorem 109: (corollary) In Euclidean Geometry, parallel lines are everywhere equidistant. (page 216)
In Euclidean Geometry, if lines K and L are parallel, and line M is a transversal that is perpendicular to lines K and L at points A and B, and line N is a transversal that is perpendicular to lines K and L at points C and D, then $AB = CD$.

Theorem 110: The Euclidean Geometry Triangle Midsegment Theorem (page 217)
In Euclidean Geometry, if the endpoints of a line segment are the midpoints of two sides of a triangle, then the line segment is parallel to the third side and is half as long as the third side. That is, a midsegment of a triangle is parallel to the third side and half as long.

Theorem 111: Properties of Medial Triangles in Euclidean Geometry (page 218)
(1) The sides of the medial triangle are parallel to sides of outer triangle and are half as long.
(2) The altitude lines of the medial triangle are the perpendicular bisectors of the sides of the outer triangle.
(3) The altitude lines of the medial triangle are concurrent.

Theorem 112: In Euclidean Geometry any given triangle is a medial triangle for some other. (page 218)

Theorem 113: (Corollary) In Euclidean Geometry, the altitude lines of any triangle are concurrent. (page 219)

Theorem 114: about n distinct parallel lines intersecting a transversal in Euclidean Geometry (page 220)
Given: in Euclidean Geometry, parallel lines $L_1, L_2, \cdots, L_n$ intersecting a transversal T at points $P_1, P_2, \cdots, P_n$ such that $P_1 * P_2 * \cdots * P_n$.

Claim: The following are equivalent
(i) Lines $L_1, L_2, \cdots, L_n$ are equally spaced parallel lines.
(ii) The lines cut congruent segments in transversal T. That is, $\overline{P_1P_2} \cong \overline{P_2P_3} \cong \cdots \cong \overline{P_{n-1}P_n}$.

Theorem 115: (Corollary) about n distinct parallel lines cutting congruent segments in transversals in Euclidean Geometry (page 220)
If a collection of n parallel lines cuts congruent segments in one transversal, then the n parallel lines must be equally spaced and so they will also cut congruent segments in any transversal.

Theorem 116: about concurrence of medians of triangles in Euclidean Geometry (page 220)
In Euclidean Geometry, the medians of any triangle are concurrent at a point that can be called the centroid. Furthermore, the distance from the centroid to any vertex is $2/3$ the length of the median drawn from that vertex.

Theorem 117: Parallel Projection in Euclidean Geometry is *one-to-one* and *onto*. (page)

Theorem 118: Parallel Projection in Euclidean Geometry preserves betweenness. (page 228)
If L, M, T are lines, and T intersects both L and M, and A, B, C are points on L with $A * B * C$, then $A' * B' * C'$.

Theorem 119: Parallel Projection in Euclidean Geometry preserves congruence of segments. (page 228)
If L, M, T are lines, and T intersects both L and M, and A, B, C, D are points on L with $\overline{AB} \cong \overline{CD}$, then $\overline{A'B'} \cong \overline{C'D'}$.

Theorem 120: Parallel Projection in Euclidean Geometry preserves ratios of lengths of segments. (page 229)
If L, M, T are lines, and T intersects both L and M, and A, B, C, D are points on L with $C \neq D$, then $\frac{A'B'}{C'D'} = \frac{AB}{CD}$.

Theorem 121: (corollary) about lines that are parallel to the base of a triangle in Euclidean Geometry. (page 229)
In Euclidean Geometry, if line T is parallel to side $\overline{BC}$ of triangle ΔABC and intersects rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points D and E, respectively, then $\frac{AD}{AB} = \frac{AE}{AC}$.

Theorem 122: The Angle Bisector Theorem. (page 230)
In Euclidean Geometry, the bisector of an angle in a triangle splits the opposite side into two segments whose lengths have the same ratio as the two other sides. That is, in ΔABC, if D is the point on side $\overline{AC}$ such that ray $\overrightarrow{BD}$ bisects angle $\angle ABC$, then $\frac{DA}{DC} = \frac{BA}{BC}$.

Theorem 123: triangle similarity is an equivalence relation (page 231)

Theorem 124: The Angle-Angle-Angle (*AAA*) Similarity Theorem for Euclidean Geometry (page 233)
If there is a one-to-one correspondence between the vertices of two triangles, and each pair of corresponding angles is a congruent pair, then the ratios of the lengths of each pair of

corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

Theorem 125: (Corollary) The Angle-Angle (*AA*) Similarity Theorem for Euclidean Geometry (page 234)
If there is a one-to-one correspondence between the vertices of two triangles, and two pairs of corresponding angles are congruent pairs, then the third pair of corresponding angles is also a congruent pair, and the ratios of the lengths of each pair of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

Theorem 126: (corollary) In Euclidean Geometry, the altitude to the hypotenuse of a right triangle creates two smaller triangles that are each similar to the larger triangle. (page 234)

Theorem 127: The Side-Side-Side (*SSS*) Similarity Theorem for Euclidean Geometry (p. 234)
If there is a one-to-one correspondence between the vertices of two triangles, and the ratios of lengths of all three pairs of corresponding sides is the same, then all three pairs of corresponding angles are congruent pairs, so the correspondence is a similarity and the triangles are similar.

Theorem 128: The Side-Angle-Side (*SAS*) Similarity Theorem for Euclidean Geometry (p. 235)
If there is a one-to-one correspondence between the vertices of two triangles, and the ratios of lengths of two pairs of corresponding sides is the same and the corresponding included angles are congruent, then the other two pairs of corresponding angles are also congruent pairs and the ratios of the lengths of all three pairs of corresponding sides is the same, so the correspondence is a similarity and the triangles are similar.

Theorem 129: About the ratios of lengths of certain line segments associated to similar triangles in Euclidean Geometry. (page 236)
In Euclidean Geometry, if $\Delta \sim \Delta'$, then

$$\frac{length\ of\ side}{length\ of\ side'} = \frac{length\ of\ altitude}{length\ of\ altitude'} = \frac{length\ of\ angle\ bisector}{length\ of\ angle\ bisector'} = \frac{length\ of\ median}{length\ of\ median'}$$

Theorem 130: The Pythagorean Theorem of Euclidean Geometry (page 237)
In Euclidean Geometry, the sum of the squares of the length of the two sides of any right triangle equals the square of the length of the hypotenuse. That is, in Euclidean Geometry, given triangle ΔABC with $a = BC$ and $b = CA$ and $c = AB$, if angle $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.

Theorem 131: The Converse of the Pythagorean Theorem of Euclidean Geometry (page 238)
In Euclidean Geometry, if the sum of the squares of the length of two sides of a triangle equals the square of the length of the third side, then the angle opposite the third side is a right angle. That is, in Euclidean Geometry, given triangle ΔABC with $a = BC$ and $b = CA$ and $c = AB$, if $a^2 + b^2 = c^2$, then angle $\angle C$ is a right angle.

Theorem 132: In Euclidean Geometry, the product of $base \cdot height$ in a triangle does not depend on which side of the triangle is chosen as the base. (page 238)

Theorem 133: (accepted without proof) Given any polygonal region, any two complexes for that region have the same area sum. (page 250)

If R is a polygonal region and C_1 and C_2 are two complexes for R, then the sum of the areas of the triangular regions of complex C_1 equals the sum of the areas of the triangular regions of complex C_2.

Theorem 134: Properties of the Area Function for Polygonal Regions (page 251)

Congruence: If R_1 and R_2 are triangular regions bounded by congruent triangles, then $Area(R_1) = Area(R_2)$.

Additivity: If R_1 and R_2 are polygonal regions whose interiors do not intersect, then $Area(R_1 \cup R_2) = Area(R_1) + Area(R_2)$.

Theorem 135: about the ratio of the areas of similar triangles (page 253)

The ratio of the areas of a pair of similar triangles is equal to the square of the ratio of the lengths of any pair of corresponding sides.

Theorem 136: polygon similarity is an equivalence relation (page 254)

Theorem 137: about the ratio of the areas of similar *n*-gons (page 256)

The ratio of the areas of a pair of similar n-gons (not necessarily convex) is equal to the square of the ratio of the lengths of any pair of corresponding sides.

Theorem 138: If two distinct arcs share both endpoints, then the sum of their arc angle measures is 360. That is, if $\widehat{ABC}$ and $\widehat{ADC}$ are distinct, then $\widehat{m}\left(\widehat{ABC}\right) + \widehat{m}\left(\widehat{ADC}\right) = 360$. (page 268)

Theorem 139: Two chords of a circle are congruent if and only if their corresponding arcs have the same measure.(page 268)

Theorem 140: The Arc Measure Addition Theorem (page 268)

If $\widehat{ABC}$ and $\widehat{CDE}$ are arcs that only intersect at C, then $\widehat{m}\left(\widehat{ACE}\right) = \widehat{m}\left(\widehat{ABC}\right) + \widehat{m}\left(\widehat{CDE}\right)$.

Theorem 141: the angle measure of an inscribed angle (Type 2) is equal to half the arc angle measure of the intercepted arc. (page 270)

Theorem 142: (Corollary) Any inscribed angle that intercepts a semicircle is a right angle. (page 272)

Theorem 143: the angle measure of an angle of Type 3. (page 272)

The angle measure of an angle of Type 3 is equal to the average of the arc angle measures of two arcs. One arc is the arc intercepted by the angle, itself. The other arc is the arc intercepted by the angle formed by the opposite rays of the original angle.

Theorem 144: the angle measure of an angle of Type 4. (page 273)

The angle measure of an angle of Type 4 is equal to one half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.)

Theorem 145: the angle measure of an angle of Type 5. (page 273)
The angle measure of an angle of Type 5 can be related to the arc angle measures of the arcs that it intersects in three useful ways:
(i) The angle measure of the angle is equal to 180 minus the arc angle measure of the smaller intercepted arc
(ii) The angle measure of the angle is equal to the arc angle measure of the larger intercepted arc minus 180
(iii) The angle measure of the angle is equal to half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.)

Theorem 146: the angle measure of an angle of Type 6. (page 274)
The angle measure of an angle of Type 6 is equal to one half the arc angle measure of the arc intercepted by the angle.

Theorem 147: the angle measure of an angle of Type 7. (page 275)
The angle measure of an angle of Type 7 is equal to one half the difference of the arc angle measures of the two arcs intercepted by the angle. (The difference computed by subracting the smaller arc angle measure from the larger one.

Theorem 148: In Euclidean Geometry, in any cyclic quadrilateral, the sum of the measures of each pair of opposite angles is 180. That is, if $Quad(ABCD)$ is cyclic, then $m(\angle A)+m(\angle C)=180$ and $m(\angle B)+m(\angle D)=180$. (page 277)

Theorem 149: about angles that intercept a given arc (page 277)
In Euclidean geometry, given an arc $\widehat{ABC}$ and a point P on the opposite side of line $\overleftrightarrow{AC}$ from point B, the following are equivalent:
(1) Point P lies on $Circle(A,B,C)$.
(2) $m(\angle APC)=\frac{\hat{m}(\widehat{ABC})}{2}$.

Theorem 150: In Euclidean Geometry, in any convex quadrilateral, if the sum of the measures of either pair of opposite angles is 180, then the quadrilateral is cyclic. (page 278)

Theorem 151: The Intersecting Secants Theorem (p. 279)
In Euclidean geometry, if two secant lines intersect, then the product of the distances from the intersection point to the two points where one secant line intersects the circle equals the product of the distances from the intersection point to the two points where the other secant line intersects the circle. That is, if secant line L passes through a point Q and intersects the circle at points A and B and secant line M passes through a point Q and intersects the circle at points D and E, then $QA \cdot QB = QA \cdot QB$.

Theorem 152: about intersecting secant and tangent lines. (page 281)
In Euclidean geometry, if a secant line and tangent line intersect, then the square of the distance from the point of intersection to the point of tangency equals the product of the distances from the intersection point to the two points where the secant line intersects the circle. That is, if a secant line passes through a point Q and intersects the circle at points A and B and a tangent line passes through Q and intersects the circle at D, then

$$(QD)^2 = QA \cdot QB$$

Theorem 153: Miquel's Theorem (page 285)
If points A, B, C, D, E, F are given such that points A, B, C are non-collinear and $A * D * B$ and $B * E * C$ and $C * F * A$, then $Circle(A, D, F)$ and $Circle(B, E, D)$ and $Circle(C, F, E)$ exist and there exists a point G that lies on all three circles.

Theorem 154: Menelaus's Theorem (page 285)
Given: points A, B, C, D, E, F such that A, B, C are non-collinear and $A * B * D$ and $B * E * C$ and $C * F * A$
Claim: The following are equivalent
(1) $F * E * D$
(2) $\frac{AD}{DB} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = -1$

Theorem 155: Ceva's Theorem (found on page 286)
Given: points A, B, C, D, E, F such that A, B, C are non-collinear and $A * D * B$ and $B * E * C$ and $C * F * A$
Claim: The following are equivalent
(1) Lines $\overleftrightarrow{AE}, \overleftrightarrow{BF}, \overleftrightarrow{CD}$ are concurrent
(2) $\frac{AD}{DB} \cdot \frac{BE}{CE} \cdot \frac{CF}{AF} = 1$

Theorem 156: (Corollary to Ceva's Theorem) (found on page 286)
If A, B, C are non-collinear points and $A * D * B$ and $B * E * C$ and $C * F * A$ are the points of tangency of the inscribed circle for ΔABC, then lines $\overleftrightarrow{AE}, \overleftrightarrow{BF}, \overleftrightarrow{CD}$ are concurrent.

Theorem 157: Collinearity of the orthcenter, centroid, and circumcenter (found on page 287)
If a non-equilateral triangle's orthcenter, centroid, and circumcenter are labeled A, B, C, Then A, B, C are collinear, with $A * B * C$ and $AB = 2BC$.

Theorem 158: Existence of a circle passing through nine special points associated to a triangle (found on page 288)
For every triangle, there exists a single circle that passes through the midpoints of the three sides, the feet of the three altitudes, and the three Euler points.

Theorem 159: The ratio $\frac{circumference}{diameter}$ is the same for all circles. (page 293)

Theorem 160: The circumference of a circle is $c = \pi d$. The area of a circle is $A = \pi r^2$. (p. 296)

Theorem 161: Function composition is associative. (page 304)
For all functions $f: A \to B$ and $g: B \to C$ and $h: C \to D$, the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal.

Theorem 162: Bijective functions have inverse functions that are also bijective. (page 308)
If $f: A \to B$ is a bijective function, then f has an inverse function $f^{-1}: B \to A$. The inverse function is also bijective.

Theorem 163: If a function has an inverse function, then both the function and its inverse are bijective. (page 309)

If functions $f: A \to B$ and $g: B \to A$ are inverses of one another (that is, if they satisfy the inverse relations), then both f and g are bijective.

Theorem 164: about the inverse of a composition of functions (page 310)

If functions $f: A \to B$ and $g: B \to C$ are both bijective, then their composition $g \circ f$ will be bijective. The inverse of the composition will be $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Theorem 165: the pair $(T, \circ)$ consisting of the set of Transformations of the Plane and the operation of composition of functions, is a group. (page 316)

Theorem 166: The composition of two isometries of the plane is also an isometry of the plane. (page 318)

Theorem 167: Every isometry of the plane is one-to-one. (page 318)

Theorem 168: For three distinct points, betweenness is related to distance between the points. (page 319)

For distinct points A, B, C, the following two statements are equivalent.

(i) $A * B * C$

(ii) $d(A, B) + d(B, C) = d(A, C)$

Theorem 169: Isometries of the plane preserve collinearity. (page 319)

If A, B, C are distinct, collinear points and f is an isometry of the plane, then $f(A), f(B), f(C)$ are distinct, collinear points.

Theorem 170: Every isometry of the plane is onto. (page 320)

Theorem 171: (corollary) Every isometry of the plane is also a transformation of the plane. (page 321)

Theorem 172: Every isometry of the plane has an inverse that is also an isometry. (page 322)

Theorem 173: the pair $(I, \circ)$ consisting of the set of Isometries of the Plane and the operation of composition of functions, is a group. (page 322)

Theorem 174: Isometries of the plane preserve lines. (page 323)

If L is a line and $f: \mathcal{P} \to \mathcal{P}$ is an isometry, then the image $f(L)$ is also a line.

Theorem 175: Isometries of the plane preserve circles. (page 324)

If $f: \mathcal{P} \to \mathcal{P}$ is an isometry, then the image of a circle is a circle with the same radius. More specifically, the image $f\big(Circle(P, r)\big)$ is the circle $Circle(f(P), r)$.

Theorem 176: If an isometry has three non-collinear fixed points, then the isometry is the identity map. (page 325)

Theorem 177: If two isometries have the same images at three non-collinear fixed points, then the isometries are in fact the same isometry. (page 325)

If $f: \mathcal{P} \rightarrow \mathcal{P}$ and $g: \mathcal{P} \rightarrow \mathcal{P}$ are isometries and A, B, C are non-collinear points such that $f(A) = g(A)$ and $f(B) = g(B)$ and $f(C) = g(C)$, then $f = g$.